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INSEPARABLE EXTENSIONS OF ALGEBRAS OVER THE STEENROD ALGEBRA WITH APPLICATIONS TO MODULAR INVARIANT THEORY OF FINITE GROUPS

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Dedicated to Clarence W. Wilkerson on the occasion of his 60th birthday

Abstract. We consider purely inseparable extensions $\mathrm{H} \hookrightarrow \sqrt[\mathcal{P}^*]{\mathrm{H}}$ of unstable Noetherian integral domains over the Steenrod algebra. It turns out that there exists a finite group $G \leq \mathrm{GL}(V)$ and a vector space decomposition $V = W_0 \oplus W_1 \oplus \cdots \oplus W_e$ such that $\overline{\mathrm{H}} = (\mathbb{F}[W_0] \otimes \mathbb{F}[W_1]^p \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e})^G$ and $\overline{\sqrt[\mathcal{P}^*]{\mathrm{H}}} = \mathbb{F}[V]^G$, where $\overline{(-)}$ denotes the integral closure. Moreover, H is Cohen-Macaulay if and only if $\sqrt[\mathcal{P}^*]{\mathrm{H}}$ is Cohen-Macaulay. Furthermore, $\overline{\mathrm{H}}$ is polynomial if and only if $\sqrt[\mathcal{P}^*]{\mathrm{H}}$ is polynomial, and $\sqrt[\mathcal{P}^*]{\mathrm{H}} = \mathbb{F}[h_1, \ldots, h_n]$ if and only if

$$\mathrm{H} = \mathbb{F}[h_1, \ldots, h_{n_0}, h_{n_0+1}^p, \ldots, h_{n_1}^p, h_{n_1+1}^{p^2}, \ldots, h_{n_e}^{p^e}],$$

where $n_e = n$ and $n_i = \dim_{\mathbb{F}}(W_0 \oplus \cdots \oplus W_i)$.

1. Introduction and outline

Let $\mathbb{K} \hookrightarrow \mathbb{L}$ be an algebraic extension of graded fields. Assume that the smaller field, $\mathbb{K}$, carries an action of the Steenrod algebra $\mathcal{P}^*$ of reduced powers. If the extension $\mathbb{K} \hookrightarrow \mathbb{L}$ is separable, then the action of $\mathcal{P}^*$ can be uniquely extended to $\mathbb{L}$. In other words, the separable closure of $\mathbb{K}$ as a field over the Steenrod algebra coincides with the separable closure in the category of graded fields; see Proposition 2.2.2 in [3] and Proposition 2.2 in [7].

If the extension, however, is purely inseparable the situation is more delicate: Let $p(X) \in \mathbb{K}[X]$ be the minimal polynomial of $l \in \mathbb{L}$. Since the extension is purely inseparable, we have that

$$p(X) = X^{p^e} - \kappa,$$

so that $l^{p^e} = \kappa$ for some $\kappa \in \mathbb{K}$. Of course, since our fields are graded, we obtain the following condition on the degrees:

$$(*) \qquad \deg(l)p^e = \deg(\kappa).$$

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However, the crucial issue is the following. If there were an extension of the $\mathcal{P}^*$-action to the larger field $\mathbb{L}$, then

$$\mathscr{P}^{\Delta_i}(\kappa) = 0 \quad \forall i$$

because by equation $(*)$ the element κ is a pth power. Thus, we need to define

$$(\star) \qquad (\mathscr{P}^i(l))^{p^e} = \mathscr{P}^{ip^e}(\kappa) \in \mathbb{K}.$$

The problem is that it does not follow that $\mathscr{P}^i(l) \in \mathbb{L}$. Nevertheless, as equation $(\star)$ shows, the inseparable closures of $\mathbb{K}$ as a graded field and as a field over the Steenrod algebra coincide. We denote this object by $\sqrt[\mathcal{P}^*]{\mathbb{K}}$.

This leads to the following question: Under which conditions can we extend the action of $\mathcal{P}^*$ from $\mathbb{K}$ to $\mathbb{L}$? Or equivalently, which intermediate fields $\mathbb{K} \subseteq \mathbb{L} \subseteq \sqrt[\mathcal{P}^*]{\mathbb{K}}$ are objects in the category of fields over the Steenrod algebra?

In this paper we study these questions in the more general framework of Noetherian integral domains H over the Steenrod algebra.

In Section 2 we recall the construction of inseparable closures over the Steenrod algebra and its basic properties. To this list we add a few more that will be of use later.

In Sections 3 and 4 we start with the investigation of inseparable extensions $\mathrm{H} \hookrightarrow \sqrt[\mathcal{P}^*]{\mathrm{H}}$, where $\sqrt[\mathcal{P}^*]{\mathrm{H}}$ is either the symmetric algebra, $\mathbb{F}[V]$, on V^* with $V = \mathbb{F}^n$, or its field of fractions, $\mathbb{F}(V)$. This has two reasons: for one, $\mathbb{F}(V)$ and $\mathbb{F}[V]$ are universal, in the sense that they are algebraically closed in our category. On the other hand, any unstable Noetherian integral domain H can be embedded into $\mathbb{F}[V]$ such that the inclusion

$$\mathrm{H} \hookrightarrow \mathbb{F}[V]$$

is finite; see the Embedding Theorem, Corollary 6.1.5 in [3]. Thus in Sections 3 and 4 we consider the diagram

$$\begin{array}{ccc} \mathrm{H} & \hookrightarrow & \sqrt[\mathcal{P}^*]{\mathrm{H}} = \mathbb{F}[V] \\ \big\downarrow & & \big\downarrow \\ \mathbb{H} & \hookrightarrow & \sqrt[\mathcal{P}^*]{\mathbb{H}} = \mathbb{F}(V), \end{array}$$

where $\mathbb{H} = FF(\mathrm{H})$ is the field of fractions of H. In Section 3 we treat the case of purely inseparable extensions of exponent one, in Section 4 we look at extensions with higher exponents e. Denote by $\overline{(-)}$ the integral closure. The results of these parts show that[1]

$$\overline{\mathrm{H}} = \mathbb{F}[W_0] \otimes \mathbb{F}[W_1]^p \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e} \quad \text{and}$$
$$\mathbb{H} = \mathbb{F}(W_0) \otimes \mathbb{F}(W_1)^p \otimes \cdots \otimes \mathbb{F}(W_e)^{p^e}$$

for some vector space decomposition $V = W_0 \oplus W_1 \oplus \cdots \oplus W_e$; see Theorem 4.13 and Corollary 4.14.[2] This reproves results in [8], Theorem II, and [3], Theorem 7.2.2. However, the proof presented here has the advantage that it gives precise information on the vector space dimensions of the W_i's.

[1] All tensor products in this manuscript are tensor products over the ground field $\mathbb{F}$.

[2] We denote by $\mathbb{F}[V]^p$ the algebra $\mathbb{F}[x_1^p, \ldots, x_n^p]$ for $\mathbb{F}[V] = \mathbb{F}[x_1, \ldots, x_n]$.

In Section 5 we come to the general case. By the Galois Embedding Theorem (Theorem 7.1.1 in [3]), we know that $\overline{\sqrt[\mathcal{P}^*]{\mathrm{H}}}$ is a ring of invariants of some finite group $G \leq \mathrm{GL}(V)$ acting linearly on $\mathbb{F}[V]$. Thus

$$\begin{array}{ccc} \overline{\mathrm{H}} & \hookrightarrow & \overline{\sqrt[\mathcal{P}^*]{\mathrm{H}}} = \mathbb{F}[V]^G \\ \downarrow & & \downarrow \\ \mathbb{H} & \hookrightarrow & \sqrt[\mathcal{P}^*]{\mathbb{H}} = \mathbb{F}(V)^G. \end{array}$$

It turns out that there exists a vector space decomposition as above,

$$V = W_0 \oplus W_1 \oplus \cdots \oplus W_e,$$

such that G acts on the flags

$$W_0 \oplus W_1 \oplus \cdots \oplus W_i$$

for all $i = 0, \ldots, e$. Moreover

$$\overline{\mathrm{H}} = (\mathbb{F}[W_0] \otimes \mathbb{F}[W_1]^p \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e})^G \quad \text{and}$$
$$\mathbb{H} = (\mathbb{F}(W_0) \otimes \mathbb{F}(W_1)^p \otimes \cdots \otimes \mathbb{F}(W_e)^{p^e})^G;$$

see Theorem 5.2. This extends results in [8], Theorem II, and [3], Theorem 7.2.2, in the sense that we are able to determine the vector space dimensions of the W_i's and, more importantly, are able to prove that the group G in question remains unchanged. In particular this means that the group G must consist of flag matrices

$$\begin{bmatrix} A_0 & 0 & & \ldots & 0 \\ * & A_1 & 0 & \ldots & 0 \\ & * & \ddots & & \vdots \\ \ldots & & \ddots & & 0 \\ * & & \ldots & * & A_e \end{bmatrix},$$

where A_i is an $m_i \times m_i$-matrix with $m_i = \dim(W_i)$. On the other hand, if V has no basis such that G consists of flag matrices, then the only purely inseparable extensions of exponent e are

$$(\mathbb{F}[V]^{p^e})^G \subseteq \mathbb{F}[V]^G.$$

In Section 6 we take a break from these constructive methods and look at homological properties of H and $\sqrt[\mathcal{P}^*]{\mathrm{H}}$. We show that H is Cohen-Macaulay if and only if $\sqrt[\mathcal{P}^*]{\mathrm{H}}$ is Cohen-Macaulay for any reduced Noetherian unstable algebra H.

This motivates Section 7, where we look at polynomial rings. It turns out that $\sqrt[\mathcal{P}^*]{\mathrm{H}}$ is a polynomial algebra if and only if H is polynomial. Moreover, $\sqrt[\mathcal{P}^*]{\mathrm{H}} = \mathbb{F}[h_1, \ldots, h_n]$ if and only if

$$\mathrm{H} = \mathbb{F}[h_1, \ldots, h_{n_0}, h_{n_0+1}^0, \ldots, h_{n_1}^p, h_{n_1+1}^{p^2}, \ldots, h_{n_e}^{p^e}],$$

where $n_e = n$ and $n_i = \dim_{\mathbb{F}}(W_0 \oplus \cdots \oplus W_i)$. Recall that an unstable $\mathcal{P}^*$-inseparably closed polynomial algebra over the Steenrod algebra is the ring of invariants $\mathbb{F}[V]^G$ for some $G \leq \mathrm{GL}(n, \mathbb{F})$ by the Galois Embedding Theorem (Theorem 7.1.1 in [3]). Combined with the results from Section 5, this means that if G consists of flag matrices, then H is polynomial if and only if $\sqrt[\mathcal{P}^*]{\mathrm{H}}$ is polynomial, and the generators are just pth powers/roots of one another. However, if G does not consist of flag matrices, then there exists no unstable algebra $\mathrm{H} \hookrightarrow \mathbb{F}[V]^G$ such that $\sqrt[\mathcal{P}^*]{\mathrm{H}} = \mathbb{F}[V]^G$. This solves a twenty-year-old conjecture due to Clarence Wilkerson; see Conjecture 5.1 in [8].

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2. Recollections and preliminaries

Let H be an unstable reduced algebra over the Steenrod algebra of reduced powers $\mathcal{P}^*$. We denote the characteristic by p, and the order of the ground field $\mathbb{F}$ by q. Recall that the Steenrod algebra contains an infinite sequence of derivations iteratively defined as

$$\mathscr{P}^{\Delta_1} = \mathscr{P}^1,$$
$$\mathscr{P}^{\Delta_i} = \mathscr{P}^{\Delta_{i-1}}\mathscr{P}^{q^{i-1}} - \mathscr{P}^{q^{i-1}}\mathscr{P}^{\Delta_{i-1}} \quad \text{for } i \geq 2.$$

We set

$$\mathscr{P}^{\Delta_0}(h) = \deg(h)h \quad \forall h \in \mathrm{H}.$$

Note that $\mathscr{P}^{\Delta_0}$ is not an element of the Steenrod algebra.

The algebra H is called $\mathcal{P}^*$-*inseparably closed*, if whenever $h \in \mathrm{H}$ and

$$\mathscr{P}^{\Delta_i}(h) = 0 \quad \forall i \geq 0,$$

then there exists an element $h' \in \mathrm{H}$ such that

$$(h')^p = h.$$

The $\mathcal{P}^*$-*inseparable closure* of H is a $\mathcal{P}^*$-inseparably closed algebra $\sqrt[\mathcal{P}^*]{\mathrm{H}}$ containing H such that the following universal property holds: Whenever we have a $\mathcal{P}^*$-inseparably closed algebra H' containing H there exists an embedding $\varphi\colon \sqrt[\mathcal{P}^*]{\mathrm{H}} \hookrightarrow \mathrm{H}'$.

The following method to construct the $\mathcal{P}^*$-inseparable closure of H is taken from Section 4.1 in [3]. Denote by $\mathcal{C} \subseteq \mathrm{H}$ the subalgebra consisting of the $\mathscr{P}^{\Delta_i}$-constants for all $i \geq 0$, i.e.,

$$\mathcal{C} = \mathcal{C}(\mathrm{H}) = \{h \in \mathrm{H} \mid \mathscr{P}^{\Delta_i}(h) = 0 \ \forall i \geq 0\}.$$

It turns out that the subalgebra of constants $\mathcal{C}$ is an unstable algebra over the Steenrod algebra (Lemma 4.1.2 loc.cit.). Moreover, it is Noetherian whenever H is (Lemma 4.1.1 loc.cit.). By construction we have integral extensions

$$\mathrm{H}^p \hookrightarrow \mathcal{C} \hookrightarrow \mathrm{H},$$

where $\mathrm{H}^p = \{h^p \mid h \in \mathrm{H}\}$. Denote by $\mathscr{S}$ a set of generators for $\mathcal{C}$ as a module over H^p. Define an algebra

$$\mathrm{H}_1 = (\mathrm{H} \otimes_{\mathbb{F}} \mathbb{F}[\gamma_s \mid s \in \mathscr{S}])/\mathcal{R}ad(\gamma_s^p - s \mid s \in \mathscr{S}),$$

where $\mathcal{R}ad(-)$ denotes the radical of the ideal $(-)$. Note that this construction comes with a canonical inclusion

$$\varphi_0 \colon \mathrm{H} \hookrightarrow \mathrm{H}_1,$$

by part (5) of Lemma 4.1.3 in [3]. Since the new algebra H_1 is again an unstable reduced algebra over the Steenrod algebra (see Lemma 4.1.4 loc.cit.), we can iterate the construction and obtain a nested sequence

$$\mathrm{H} = \mathrm{H}_0 \hookrightarrow \mathrm{H}_1 \hookrightarrow \cdots \hookrightarrow \mathrm{H}_i \hookrightarrow \cdots \hookrightarrow$$

of unstable reduced algebras over the Steenrod algebra. The colimit of this sequence is the $\mathcal{P}^*$-inseparable closure of H (Proposition 4.1.5 loc.cit.). We recall the basic properties of $\sqrt[\mathcal{P}^*]{\mathrm{H}}$ and the intermediate algebras H_i.

Proposition 2.1. *Consider the chain of unstable reduced algebras*

$$\mathrm{H} = \mathrm{H}_0 \hookrightarrow \mathrm{H}_1 \hookrightarrow \cdots \hookrightarrow \mathrm{H}_i \hookrightarrow \cdots \hookrightarrow \sqrt[\mathcal{P}^*]{\mathrm{H}}.$$

Then the following statements hold.

(1) *If one of the algebras in this chain is an integral domain, then so are the others.*
(2) $\mathrm{H} \hookrightarrow \sqrt[\mathcal{P}^*]{\mathrm{H}}$ *is an integral extension, and both algebras have the same Krull dimension.*
(3) *If* H *is integrally closed, then so is* $\sqrt[\mathcal{P}^*]{\mathrm{H}}$.
(4) *The following statements are equivalent.*
   - H *is Noetherian.*
   - H_i *is Noetherian.*
   - $\sqrt[\mathcal{P}^*]{\mathrm{H}}$ *is Noetherian.*
   - *There exists an* r *such that*

$$\mathrm{H}_r = \mathrm{H}_{r+1} = \cdots = \sqrt[\mathcal{P}^*]{\mathrm{H}}.$$

Proof. For (1)–(3) see Proposition 4.2.1 in [3]. For (4) see part (2) of Lemma 4.1.3, Lemma 4.2.2, Proposition 4.2.4, and Theorem 6.3.1 loc.cit. □

Lemma 2.2. *Let* H *be an integral domain. If* H *is integrally closed, then the algebras* H_i *are also integrally closed for all* i.

Proof. It is shown in part (5) of Proposition 4.2.1 in [3] that $\sqrt[\mathcal{P}^*]{\mathrm{H}}$ is integrally closed whenever H is integrally closed. The same argument presented there can be used to show that also the algebras H_i are also integrally closed. □

In the same way the $\mathcal{P}^*$-inseparable closure of a field $\mathbb{K}$ over the Steenrod algebra can be constructed. So we obtain a chain of fields over the Steenrod algebra

$$\mathbb{K} = \mathbb{K}_0 \hookrightarrow \mathbb{K}_1 \hookrightarrow \cdots \hookrightarrow \mathbb{K}_i \hookrightarrow \cdots \hookrightarrow \sqrt[\mathcal{P}^*]{\mathbb{K}}$$

by adjoining successively pth roots. Again the pth powers are detected by the vanishing of the derivations $\mathscr{P}^{\Delta_i}$; see Section 2.3 in [3].

Let H be an unstable integral domain over the Steenrod algebra. Denote by $\mathbb{H}$ its field of fractions. We have seen in Proposition 4.2.6 in [3] that

$$FF(\sqrt[\mathcal{P}^*]{\mathrm{H}}) = \sqrt[\mathcal{P}^*]{\mathbb{H}},$$

where $FF(-)$ denotes the field of fraction functor.

Our first goal is to refine this statement. For this we need the following result.

Proposition 2.3. *Let* H *be an unstable integral domain. Then*

$$\mathcal{C}(\mathbb{H}) = FF(\mathcal{C}(\mathrm{H})).$$

Proof. Let $\frac{f_1}{f_2} \in \mathcal{C}(\mathbb{H})$, $f_1, f_2 \in \mathrm{H}$. Then there exists an element $\frac{h_1}{h_2} \in \sqrt[\mathcal{P}^*]{\mathbb{H}} = FF(\sqrt[\mathcal{P}^*]{\mathrm{H}})$, $h_1 h_2 \in \sqrt[\mathcal{P}^*]{\mathrm{H}}$, such that

$$\frac{h_1^{p^k}}{h_2^{p^k}} = \frac{f_1}{f_2}$$

for some $k \in \mathbb{N}_0$. Furthermore, since $h_1, h_2 \in \sqrt[\mathcal{P}^*]{\mathrm{H}}$ we can choose k such that $h_1^{p^k}, h_2^{p^k} \in \mathrm{H}$. By construction, $h_1^{p^k}, h_2^{p^k}$ are in the subalgebra of constants, $\mathcal{C}(\mathrm{H})$. Thus

$$\frac{f_1}{f_2} = \frac{h_1^{p^k}}{h_2^{p^k}} \in FF(\mathcal{C}(\mathrm{H}))$$

which shows that

$$\mathcal{C}(\mathbb{H}) \subseteq FF(\mathcal{C}(\mathrm{H})).$$

Conversely, let $\frac{f_1}{f_2} \in FF(\mathcal{C}(\mathrm{H}))$ with $f_1, f_2 \in \mathcal{C}(\mathrm{H})$. Then

$$\frac{f_1}{f_2} \in \mathbb{H}$$

is a constant because

$$\mathscr{P}^{\Delta_i}\left(\frac{f_1}{f_2}\right) = \frac{\mathscr{P}^{\Delta_i}(f_1)f_2 - f_1\mathscr{P}^{\Delta_i}(f_2)}{f_2^2} = 0$$

for all $i \in \mathbb{N}_0$. □

Proposition 2.4. *Let* H *be an unstable integral domain over the Steenrod algebra. Denote by* $\mathbb{H}$ *its field of fractions. Then for all* $i \in \mathbb{N}_0$ *we have*

$$FF(\mathrm{H}_i) = \mathbb{H}_i.$$

Proof. By induction it is enough to show the statement for $i = 1$. If $\frac{h_1}{h_2} \in FF(\mathrm{H})_1$ for $h_1, h_2 \in \mathrm{H}_1$, then $\frac{h_1^p}{h_2^p} \in FF(\mathrm{H}_0) = \mathbb{H}_0 = \mathbb{H}$. Thus $\frac{h_1}{h_2} \in \mathbb{H}_1$, since $\mathbb{H}_1$ is obtained from $\mathbb{H}_0$ by adjoining all pth roots. Thus $FF(\mathrm{H}_1) \subseteq \mathbb{H}_1$.

We prove the reverse inclusion. Let $h \in \mathbb{H}_1$. Then $h^p \in \mathbb{H} = FF(\mathrm{H})$. Thus by Proposition 2.3

$$h^p \in \mathcal{C}(\mathbb{H}) = FF(\mathcal{C}(\mathrm{H})).$$

Thus there exist elements $h_1, h_2 \in \mathcal{C}(\mathrm{H})$ such that

$$h^p = \frac{h_1}{h_2}.$$

Moreover, since the elements h_1, h_2 are constants they have pth roots, say f_1, f_2, in H_1. Thus

$$h = \frac{f_1}{f_2} \in FF(\mathrm{H}_1)$$

and we are done. □

Hence we obtain chains

$$\begin{array}{ccccccccc}
\mathrm{H} = \mathrm{H}_0 & \hookrightarrow & \mathrm{H}_1 & \hookrightarrow \cdots \hookrightarrow & \mathrm{H}_i & \hookrightarrow \cdots \hookrightarrow & \sqrt[\mathcal{P}^*]{\mathrm{H}} \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
FF(\mathrm{H}_0) & \hookrightarrow & FF(\mathrm{H}_1) & \hookrightarrow \cdots \hookrightarrow & FF(\mathrm{H}_i) & \hookrightarrow \cdots \hookrightarrow & FF(\sqrt[\mathcal{P}^*]{\mathrm{H}}) \\
\| & & \| & & \| & & \| \\
\mathbb{H}_0 & \hookrightarrow & \mathbb{H}_1 & \hookrightarrow \cdots \hookrightarrow & \mathbb{H}_i & \hookrightarrow \cdots \hookrightarrow & \sqrt[\mathcal{P}^*]{\mathbb{H}}.
\end{array}$$

Let H be an unstable Noetherian integral domain over the Steenrod algebra. Then there exists an $r \in \mathbb{N}_0$ such that $\mathrm{H}_r = \sqrt[\mathcal{P}^*]{\mathrm{H}}$; see Theorem 6.1.3 and Proposition 4.2.4 in [3]. Also, there exists an $s \in \mathbb{N}_0$ such that $\mathbb{H}_s = \sqrt[\mathcal{P}^*]{\mathbb{H}}$, loc.cit. Without loss of generality we assume that r and s are minimal with respect to this property. Then by Proposition 4.2.4 in [3] we have that $r \geq s$. Thus for Noetherian unstable algebras we obtain *finite* chains

$$\begin{array}{ccccccccccc}
\mathrm{H} = \mathrm{H}_0 & \hookrightarrow & \mathrm{H}_1 & \hookrightarrow \cdots \hookrightarrow & \mathrm{H}_s & \hookrightarrow & \mathrm{H}_{s+1} & \hookrightarrow \cdots \hookrightarrow & \mathrm{H}_r = \sqrt[\mathcal{P}^*]{\mathrm{H}} \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
FF(\mathrm{H}_0) & \hookrightarrow & FF(\mathrm{H}_1) & \hookrightarrow \cdots \hookrightarrow & FF(\mathrm{H}_s) & = & FF(\mathrm{H}_{s+1}) & = \cdots = & FF(\mathrm{H}_r) \\
\| & & \| & & \| & & \| & & \| \\
\mathbb{H}_0 & \hookrightarrow & \mathbb{H}_1 & \hookrightarrow \cdots \hookrightarrow & \mathbb{H}_s & = & \mathbb{H}_{s+1} & = \cdots = & \mathbb{H}_r = \sqrt[\mathcal{P}^*]{\mathbb{H}}.
\end{array}$$

Corollary 2.5. *Let* H *be an unstable Noetherian integral domain. Let* H *be integrally closed. Then with the preceding notation,* $r = s$*.*

Proof. By Proposition 2.4, $FF(\mathrm{H}_i) = \mathbb{H}_i$. Moreover, H_i is integrally closed for all i by Lemma 2.2. Thus for all i the unstable part of $FF(\mathrm{H}_i)$ is

$$\mathcal{U}n(FF(\mathrm{H}_i)) = \mathrm{H}_i;$$

see Theorem 2.4 in [4]. Thus

$$\mathrm{H}_s = \mathcal{U}n(FF(\mathrm{H}_s)) = \mathcal{U}n(FF(\mathrm{H}_r)) = \mathrm{H}_r$$

as desired. □

3. Inseparable extensions of exponent 1

Let H be an unstable Noetherian reduced algebra over the Steenrod algebra. Define the H-module

$$\mathrm{Der}_{\mathrm{H}} = \mathrm{span}_{\mathrm{H}}\{\mathscr{P}^{\Delta_i} \mid i \in \mathbb{N}_0\}.$$

Then $\mathrm{Der}_{\mathrm{H}}$ is free as a module over H; see Proposition 1.1.7 and Theorem 1.2.1 in [3].[3] Moreover it is a restricted Lie algebra of derivations acting on H; cf. Section 2.4 in [3]. We denote by

$$\mathcal{C}_{\mathrm{Der}_{\mathrm{H}}}(\mathrm{H}) = \{h \in \mathrm{H} \mid \mathscr{P}^{\Delta_i}(h) = 0 \ \forall i\} \subseteq \mathrm{H}$$

the subalgebra of constants with respect to the derivations in $\mathrm{Der}_{\mathrm{H}}$.

[3]The module $\mathrm{Der}_{\mathrm{H}}$ is the module $\Delta(\mathrm{H})$ in this reference.

Remark. In Section 2 we called the subalgebra of constants just $\mathcal{C} = \mathcal{C}(\mathrm{H})$. For what follows however, we need to keep track of the module of derivations that is used.

Clearly,

$$\mathrm{H}^p \subseteq \mathcal{C}_{\mathrm{Der}_{\mathrm{H}}}$$

as $\mathscr{P}^{\Delta_i}(h^p) = 0$ for all $h \in \mathrm{H}$ and $i \in \mathbb{N}_0$. Since the extension

$$\mathrm{H}^p \subseteq \mathrm{H}$$

is purely inseparable of exponent one, so is the extension

$$\mathrm{H}^p \subseteq \mathcal{C}_{\mathrm{Der}_{\mathrm{H}}}(\mathrm{H}).$$

Thus

$$\sqrt[\mathcal{P}^*]{\mathrm{H}^p} = \sqrt[\mathcal{P}^*]{\mathcal{C}_{\mathrm{Der}_{\mathrm{H}}}(\mathrm{H})} = \sqrt[\mathcal{P}^*]{\mathrm{H}}.$$

Proposition 3.1. *Let* H *be an unstable reduced algebra over the Steenrod algebra. Then* H *is* $\mathcal{P}^*$*-inseparably closed if and only if*

$$\mathcal{C}_{\mathrm{Der}_{\mathrm{H}}}(\mathrm{H}) = \mathrm{H}^p.$$

Proof. Assume that H is $\mathcal{P}^*$-inseparably closed. By what we have done so far we know that $\mathrm{H}^p \subseteq \mathcal{C}_{\mathrm{Der}_{\mathrm{H}}}(\mathrm{H})$. To prove the reverse inclusion, let $h \in \mathcal{C}_{\mathrm{Der}_{\mathrm{H}}}(\mathrm{H})$. Then $\mathscr{P}^{\Delta_i}(h) = 0$ for all $i \in \mathbb{N}_0$. Thus there exists an element

$$f \in \sqrt[\mathcal{P}^*]{\mathcal{C}_{\mathrm{Der}_{\mathrm{H}}}(\mathrm{H})} = \sqrt[\mathcal{P}^*]{\mathrm{H}} = \mathrm{H}$$

such that $f^{p^k} = h$. Since $f^{p^k} \in \mathrm{H}^p$ for all $f \in \mathrm{H}$, we have $h = f^{p^k} \in \mathrm{H}^p$, and hence $\mathrm{H}^p = \mathcal{C}_{\mathrm{Der}_{\mathrm{H}}}(\mathrm{H})$. Conversely, assume that

$$\mathrm{H}^p = \mathcal{C}_{\mathrm{Der}_{\mathrm{H}}}(\mathrm{H}) \subseteq \mathrm{H} \subseteq \sqrt[\mathcal{P}^*]{\mathrm{H}}.$$

Let $h \in \sqrt[\mathcal{P}^*]{\mathrm{H}}$. Thus there exists a $k \in \mathbb{N}_0$ such that $h^{p^k} \in \mathrm{H}$. Since $\mathrm{Der}_{\mathrm{H}}$ vanishes on pth powers we find that

$$h^{p^k} \in \mathcal{C}_{\mathrm{Der}_{\mathrm{H}}}(\mathrm{H}) = \mathrm{H}^p.$$

Thus $h^{p^{k-1}} \in \mathrm{H}$. Iteratively we find that $h \in \mathrm{H}$, i.e., H is $\mathcal{P}^*$-inseparably closed. □

Corollary 3.2. *We have*

$$\mathcal{C}_{\mathrm{Der}_{\mathbb{F}[V]}}(\mathbb{F}[V]) = \mathbb{F}[V]^p = \mathbb{F}[x_1^p, \ldots, x_n^p].$$

Proof. Since $\mathbb{F}[V]$ is $\mathcal{P}^*$-inseparably closed (see Corollary 4.2.8 in [3]), this result is an immediate corollary of Proposition 3.1. □

We recall some facts about $\mathrm{Der}_{\mathrm{H}}$ and its action on H. First the Lie algebra structure is particularly simple, namely

$$(\ddagger) \qquad [\mathscr{P}^{\Delta_i}, \mathscr{P}^{\Delta_j}] = \begin{cases} 0 & \text{if } i, j > 0, \\ \mathscr{P}^{\Delta_i} & \text{if } i \neq 0 \text{ and } j = 0, \\ -\mathscr{P}^{\Delta_j} & \text{if } i = 0 \text{ and } j \neq 0 \end{cases}$$

(see the remark on page 12 of [3]), and

$$(✠) \qquad (\mathscr{P}^{\Delta_i})^p = 0$$

(see Section 2.4 in [3]). The Δ-*length* of H is defined to be the smallest integer[4] λ_{H} such that the derivation

$$h_0 \mathscr{P}^{\Delta_{i_0}} + \cdots + h_\lambda \mathscr{P}^{\Delta_{i_\lambda}}$$

vanishes on H for some $h_0, \ldots, h_\lambda \in \mathrm{H}$ and $i_0, \ldots, i_\lambda \in \mathbb{N}_0$ (see Section 1.2 in [3]).[5] In this case, any $\lambda + 1$ derivations are linearly dependent over H (Proposition 1.1.7 in [3]). The Δ-length λ_{H} is at most the Krull dimension of H over $\mathbb{F}$ (cf. Corollary 1.2.2 in [3]). Moreover, the coefficients can be chosen to be

$$h_i = (-1)^i \mathbf{d}_{\lambda,i}$$

(up to a sign) the Dickson classes in dimension λ (see Theorems 5.1.9 and 5.2.1 in [3]). Note that by convention $\mathbf{d}_{\lambda,\lambda} = 1$. Then the *normalized* equation

$$(\mathbf{d}_{\lambda,0} \mathscr{P}^{\Delta_0} - \cdots + (-1)^\lambda \mathbf{d}_{\lambda,\lambda} \mathscr{P}^{\Delta_\lambda})(h) = 0 \quad \forall h \in \mathrm{H}$$

is called *the* Δ-*relation* of H. By abuse of notation, we also call the element

$$\mathbf{d}_{\mathrm{H}} = \mathbf{d}_{\lambda,0} \mathscr{P}^{\Delta_0} - \cdots + (-1)^\lambda \mathbf{d}_{\lambda,\lambda} \mathscr{P}^{\Delta_\lambda} \in \mathrm{Der}_{\mathrm{H}}$$

the Δ-*relation* for H.[6] Finally we note that the Δ-length of H is equal to its Krull dimension if H is $\mathcal{P}^*$-inseparably closed (cf. Theorem 8.1.5 in [3]). The converse is not quite true as the following example taken from Section 7.4 in [3] shows.

Example 3.3. Consider the field $\mathbb{F} = \mathbb{F}_2$ with two elements, and take a polynomial algebra in two linear generators $x, y, \mathbb{F}[x, y]$. The Dickson algebra in this case is

$$\mathcal{D}(2) = \mathbb{F}[x^2y + xy^2, x^2 + y^2 + xy] \hookrightarrow \mathbb{F}[x, y].$$

Define an intermediate algebra H by

$$\mathrm{H} = \mathbb{F}[x^2 + y^2, xy, xy(x + y)]/((x^2 + y^2)(xy) + (xy(x + y))^2).$$

Then H is an unstable integral domain, but it is not $\mathcal{P}^*$-inseparably closed because

$$\mathscr{P}^{\Delta_i}(x^2 + y^2) = 0 \quad \forall i \geq 0, \text{ but } x + y \notin \mathrm{H}.$$

However, its Δ-relation

$$\mathbf{d}_{\mathrm{H}} = \mathbf{d}_{2,0} \mathscr{P}^{\Delta_0} - \mathbf{d}_{2,1} \mathscr{P}^{\Delta_1} + \mathscr{P}^{\Delta_2}$$

has length 2, which is equal to its Krull dimension. Note that the field of fractions of H,

$$FF(\mathrm{H}) = \mathbb{F}(x + y, xy),$$

is inseparably closed. Therefore the algebra H is not integrally closed because $x + y \notin \mathrm{H}$ (cf. Corollary 2.5).

Proposition 3.4. *Let* H *be an unstable Noetherian integral domain. If the* Δ*-length* λ_{H} *is equal to the Krull dimension* n *of* H*, then*

$$\sqrt[\mathcal{P}^*]{\mathrm{H}} \subseteq \overline{\mathrm{H}},$$

where $\overline{(-)}$ *denotes the integral closure.*

---

[4]If there is no possible confusion we will omit the subscript and just write $\lambda = \lambda_{\mathrm{H}}$.

[5]If $\lambda_{\mathrm{H}} \in \mathbb{N}_0$ exists, then H is called Δ-*finite*. This is a weaker condition than Noetherianess. For example the polynomial algebra $\mathbb{F}[x_1^p, x_2^p, \ldots]$ in infinitely many generators has Δ-length zero, but it is not Noetherian.

[6]We will suppress the subscript, and write $\mathbf{d}$ for $\mathbf{d}_{\mathrm{H}}$ if no confusion is possible.

Proof. Since the Δ-length of H is equal to its Krull dimension, we have integral extensions

$$\mathcal{D}(n) \hookrightarrow \mathrm{H} \hookrightarrow \sqrt[\mathcal{P}^*]{\mathrm{H}} \hookrightarrow \mathbb{F}[V]$$

by the Little Imbedding Theorem (Theorem 7.4.4 in [3]) and the Embedding Theorem (Corollary 6.1.5 loc.cit.). The corresponding extensions of the field of fractions are Galois extensions, so in particular separable. Since $FF(\mathrm{H}) \subseteq FF(\sqrt[\mathcal{P}^*]{\mathrm{H}})$ is also purely inseparable, we have $FF(\sqrt[\mathcal{P}^*]{\mathrm{H}}) = FF(\mathrm{H}) = FF(\overline{\mathrm{H}})$. Thus $FF(\mathrm{H}) = \mathbb{F}(V)^G$ for some group $G \leq \mathrm{GL}(V)$. Therefore, $\overline{\mathrm{H}} = \mathbb{F}[V]^G$ is inseparably closed. Hence $\sqrt[\mathcal{P}^*]{\mathrm{H}} \hookrightarrow \overline{\mathrm{H}}$ by the universal property of the inseparable closure. □

Remark. Note that it follows from the preceding result that if H is integrally closed and the Δ-length is equal to its Krull dimension, then

$$\sqrt[\mathcal{P}^*]{\mathrm{H}} = \overline{\mathrm{H}} = \mathrm{H}.$$

We want to investigate purely inseparable extensions $\mathrm{H} \hookrightarrow \mathbb{F}[V]$ of exponent one, i.e., we have

$$\mathbb{F}[V]^p \hookrightarrow \mathrm{H} \hookrightarrow \mathbb{F}[V].$$

For this we turn our attention to the corresponding extensions of fields of fractions

$$\mathbb{F}(V)^p \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{F}(V).$$

Let $\mathrm{Der}_{\mathbb{H}}$ be the vector space over $\mathbb{H}$ generated by the elements $\mathscr{P}^{\Delta_i}$ for $i \in \mathbb{N}_0$. Since the relations (✠) and (‡) are intrinsic of the Steenrod algebra, the vector space $\mathrm{Der}_{\mathbb{H}}$ is also a restricted Lie algebra. Thus any vector subspace of $\mathrm{Der}_{\mathbb{H}}$ is a restricted Lie subalgebra and vice versa.

Proposition 3.5. *Let* H *be an unstable integral domain and* $\mathbb{H}$ *its field of fractions. The vector space* $\mathrm{Der}_{\mathbb{H}}$ *satisfies the following properties.*

(1) *The elements* $\mathscr{P}^{\Delta_i}$ *are derivations on* $\mathbb{H}$.
(2) *The* Δ*-relation* $\mathbb{H}$ *is well defined, and coincides with the* Δ*-relation on* H. *In particular, the* Δ*-lengths are equal.*

Proof. AD(1): The action of $\mathscr{P}^{\Delta_i}$ on $\mathbb{H}$ is given by the formula

$$\mathscr{P}^{\Delta_i}\left(\frac{f_1}{f_2}\right) = \frac{\mathscr{P}^{\Delta_i}(f_1)f_2 - f_1\mathscr{P}^{\Delta_i}(f_2)}{f_2^2}$$

for any $f_1, f_2 \in \mathrm{H}$. Thus they are derivations on the field of fractions also.

AD(2): The set $\mathrm{Der}_{\mathbb{H}}$ is a vector space by construction. Let $\lambda_{\mathbb{H}}$ be its dimension. Then any $\lambda_{\mathbb{H}} + 1$ elements are linearly independent. Thus the Δ-length is $\lambda_{\mathbb{H}}$ with Δ-relation

$$d = f_0\mathscr{P}^{\Delta_0} + \cdots + f_\lambda \mathscr{P}^{\Delta_\lambda}.$$

Without loss of generality we can assume that the coefficients $f_i \in \mathrm{H}$ for all i. Thus $\lambda_{\mathbb{H}}$ is at least equal to the Δ-length, λ_{H}, of H. On the other hand, if $\mathbf{d}_{\mathrm{H}}$ is a Δ-relation for H, then by

$$\mathbf{d}_{\mathrm{H}}\left(\frac{f_1}{f_2}\right) = \frac{\mathbf{d}_{\mathrm{H}}(f_1)f_2 - f_1\mathbf{d}_{\mathrm{H}}(f_2)}{f_2^2} = 0$$

$\mathbf{d}_{\mathrm{H}}$ vanishes also on $\mathbb{H}$. Thus $\lambda_{\mathbb{H}} \leq \lambda_{\mathrm{H}}$. Therefore $\lambda_{\mathbb{H}} = \lambda_{\mathrm{H}}$ and $\mathbf{d}_{\mathbb{H}} = \mathbf{d}_{\mathrm{H}}$. □

Corollary 3.6. *Let* H *be an unstable integral domain and* $\overline{\mathrm{H}}$ *its integral closure. Then*

$$\lambda_{\mathrm{H}} = \lambda_{\overline{\mathrm{H}}} \quad \textit{and} \quad \mathbf{d}_{\mathrm{H}} = \mathbf{d}_{\overline{\mathrm{H}}}.$$

Proof. By Proposition 3.5, part (2), Δ-lengths, as well as the Δ-relations of H and its field of fractions, coincide. Since H and $\overline{\mathrm{H}}$ have the same field of fractions we are done. □

Lemma 3.7. *Let* $\mathrm{H}' \subseteq \mathrm{H}$ *be unstable reduced Noetherian algebras over the Steenrod algebra. Denote by* $\lambda_{\mathrm{H}'}$*, resp.* λ_{H}*, the* Δ*-length of* H'*, resp.* H*. Then* $\lambda_{\mathrm{H}'} \leq \lambda_{\mathrm{H}}$*.*

Proof. By Corollary 3.6 the Δ-length and Δ-relation of an unstable algebra H and its integral closure $\overline{\mathrm{H}}$ are equal. Thus without loss of generality we assume that H' and H are integrally closed.

Denote by $\mathcal{D}(l)$ the Dickson algebra of dimension l. By Theorem 5.1.9 in [3]

$$\mathcal{D}(\lambda_{\mathrm{H}}) \hookrightarrow \mathbb{H}$$

is a maximal Dickson algebra in $\mathbb{H}$. Applying the same theorem for H' gives

$$\mathcal{D}(\lambda_{\mathrm{H}'}) \hookrightarrow \mathbb{H}' \hookrightarrow \mathbb{H}.$$

Since $\mathcal{D}(\lambda_{\mathrm{H}})$ is the maximal Dickson algebra in $\mathbb{H}$, we find that $\lambda_{\mathbb{H}'} \leq \lambda_{\mathbb{H}}$ as desired. □

Lemma 3.8. *Let* U *and* W *be finite dimensional vector spaces over* $\mathbb{F}$*. We note that the* Δ*-length* λ *of* $\mathbb{F}[U] \otimes \mathbb{F}[W]^{p^t}$*,* $t > 0$*, is equal to the vector space dimension of* U *with* Δ*-relation*

$$\mathbf{d} = \mathbf{d}_{\lambda,0}\mathscr{P}^{\Delta_0} - \cdots + (-1)^{\lambda}\mathbf{d}_{\lambda,\lambda}\mathscr{P}^{\Delta_\lambda},$$

where $\mathbb{F}[U]^{\mathrm{GL}(\lambda,\mathbb{F})} = \mathbb{F}[\mathbf{d}_{\lambda,0}, \ldots, \mathbf{d}_{\lambda,\lambda-1}]$*.*

Proof. The element $\mathbf{d}$ is a Δ-relation for $\mathbb{F}[U]$ by Theorem 1.2.3 in [3]. Since $\mathscr{P}^{\Delta_i}$ vanishes on pth powers for all $i \in \mathbb{N}_0$, the element $\mathbf{d}$ vanishes on $\mathbb{F}[U] \otimes \mathbb{F}[W]^{p^t}$. So, $\lambda \leq \dim_{\mathbb{F}}(U)$.

On the other hand, $\mathbb{F}[U] \hookrightarrow \mathbb{F}[U] \otimes \mathbb{F}[W]^{p^t}$. Therefore by Lemma 3.7 the Δ-length is at least $\dim_{\mathbb{F}}(U)$, and we are done. □

Corollary 3.9. *The* Δ*-length* λ *of* $\mathbb{F}(U) \otimes \mathbb{F}(W)^{p^t}$ *is equal to the vector space dimension of* U*, for* $t \geq 1$*. The subfield of constants is*

$$\mathcal{C}_{\mathrm{Der}_{\mathbb{F}(U)\otimes\mathbb{F}(W)^p}}(\mathbb{F}(U) \otimes \mathbb{F}(W)^{p^t}) = \mathbb{F}(U)^p \otimes \mathbb{F}(W)^{p^t}.$$

Moreover, the Δ*-relation is*

$$\mathbf{d} = \mathbf{d}_{\lambda,0}\mathscr{P}^{\Delta_0} + \cdots + (-1)^{\lambda}\mathbf{d}_{\lambda,\lambda}\mathscr{P}^{\Delta_\lambda}.$$

Proof. This is immediate from part (2) of Proposition 3.5, Lemma 3.8, and Corollary 3.2. □

Since the Δ-relation of $\mathbb{F}(V)$ has length $n = \dim_{\mathbb{F}}(V)$, we have $\dim_{\mathbb{F}(V)}(\mathrm{Der}_{\mathbb{F}(V)}) = n$ and

$$\mathrm{Der}_{\mathbb{F}(V)} = \mathrm{span}_{\mathbb{F}(V)}\{\mathscr{P}^{\Delta_0}, \ldots, \mathscr{P}^{\Delta_{n-1}}\}.$$

Moreover, the index over the subfield of constants is

$$[\mathbb{F}(V) : \mathbb{F}(V)^p] = p^n.$$

Thus we can apply the structure theorem for purely inseparable extensions of exponent one (see, e.g., Chapter IV, Section 8 in [1]). It tells us that

$$\mathbb{H} \subseteq \mathbb{F}(V)$$

is a purely inseparable extension of exponent one if and only if there exists a restricted Lie subalgebra $\mathrm{D} \subseteq \mathrm{Der}_{\mathbb{F}(V)}$ such that

$$\mathbb{H} = \mathcal{C}_{\mathrm{D}}(\mathbb{F}(V)).$$

So, take a subspace $\mathrm{D} \subseteq \mathrm{Der}_{\mathbb{F}(V)}$. We recall from Corollary 3.9 that

$$\mathbb{F}(V) = \mathcal{C}_{\mathrm{D}}(\mathbb{F}(V)) \quad \text{for } \mathrm{D} = 0$$

and

$$\mathbb{F}(V)^p = \mathcal{C}_{\mathrm{D}}(\mathbb{F}(V)) \quad \text{for } \mathrm{D} = \mathrm{Der}_{\mathbb{F}(V)}\,.$$

Thus we are left to characterize those $\mathrm{D} \subseteq \mathrm{Der}_{\mathbb{F}(V)}$ such that $\mathbb{H} = \mathcal{C}_{\mathrm{D}}(\mathbb{F}(V))$ carries a $\mathcal{P}^*$-module structure.

Proposition 3.10. *Let $\mathbb{K} \hookrightarrow \mathbb{F}(V)$ be a field over the Steenrod algebra. Let $\mathbf{d}$ be the Δ-relation of $\mathbb{K}$ with Δ-length λ. Then the vector space of derivations vanishing on $\mathbb{K}$,*

$$\mathrm{D}_{\mathbb{K}} = \mathrm{span}_{\mathbb{F}}\{d \in \mathrm{Der}_{\mathbb{F}(V)} \, |d|_{\mathbb{K}} = 0\},$$

has dimension $n - \lambda$, where $n = \dim_{\mathbb{F}}(V)$.

Proof. The Δ-relation of $\mathbb{K}$ is

$$\mathbf{d} = \mathbf{d}_{\lambda,0}\mathscr{P}^{\Delta_0} + \cdots + (-1)^{\lambda}\mathbf{d}_{\lambda,\lambda}\mathscr{P}^{\Delta_\lambda}.$$

By Proposition 1.1.7 in [3] any $\lambda + 1$ derivations in $\mathrm{Der}_{\mathbb{F}(V)}$ are linearly dependent. Moreover by Lemma 1.1.8 loc.cit. we find that in particular the $n - \lambda$ elements

$$\mathbf{d}_i = \mathbf{d}_{\lambda,0}^{q^i}\mathscr{P}^{\Delta_i} + \cdots + (-1)^{\lambda}\mathbf{d}_{\lambda,\lambda}^{q^i}\mathscr{P}^{\Delta_{\lambda+1}}$$

for $i = 0, \ldots, n - \lambda - 1$ vanish on $\mathbb{K}$. Since the $\mathbf{d}_i$'s are linearly independent in $\mathrm{Der}_{\mathbb{F}(V)}$ we have that

$$\dim(\mathrm{D}_{\mathbb{K}}) \geq n - \lambda.$$

On the other hand, if $d \in \mathrm{D}_{\mathbb{K}}$, then

$$d = f_0\mathscr{P}^{\Delta_0} + \cdots + f_{n-1}\mathscr{P}^{\Delta_{n-1}}.$$

Then there are $k_0, \ldots, k_{n-1-\lambda}$ such that

$$(*) \qquad d - \sum_{i=0}^{n-1-\lambda} k_i\mathbf{d}_i = f_0'\mathscr{P}^{\Delta_0} + \cdots + f_{\lambda-1}'\mathscr{P}^{\Delta_{\lambda-1}}$$

for some $f_0', \ldots, f_{\lambda-1}' \in \mathbb{K}$. Thus if d were linearly independent of the $\mathbf{d}_i$'s, then the expression $(*)$ is not zero. This in turn means that there is a relation on $\mathbb{K}$ shorter than the Δ-relation. This is a contradiction. Therefore $\dim(\mathrm{D}_{\mathbb{K}}) = n - \lambda$. □

Theorem 3.11. *The extension $\mathbb{H} \subseteq \mathbb{F}(V)$ is a purely inseparable extension of exponent one of fields over the Steenrod algebra if and only if*

$$\mathbb{H} = \mathbb{F}(x_1, \ldots, x_k, x_{k+1}^p, \ldots, x_n^p) = \mathbb{F}(U) \otimes \mathbb{F}(V/U)^p$$

for some $k \in \{1, \ldots, n\}$ and $\dim(U) = k$. Furthermore, in this case

$$\mathbb{H} = \mathcal{C}_{\mathrm{D}}(\mathbb{F}(V))$$

where D has vector space dimension $n - k$. If $k < n$, then D is generated by the Δ-relation of $\mathbb{H}$,

$$\mathbf{d}_{\mathbb{H}} = \mathbf{d}_{k,0}\mathscr{P}^{\Delta_0} + \cdots + (-1)^k\mathbf{d}_{k,k}\mathscr{P}^{\Delta_k}$$

and its translates

$$\mathbf{d}_i = \mathbf{d}_{k,0}^{q^i}\mathscr{P}^{\Delta_i} + \cdots + (-1)^k \mathbf{d}_{k,k}^{q^i}\mathscr{P}^{\Delta_{k+i}}$$

for $i = 1, \ldots, n-k-1$.

Proof. If

$$\mathbb{H} = \mathbb{F}(x_1, \ldots, x_k, x_{k+1}^p, \ldots, x_n^p),$$

then it is clearly a field over the Steenrod algebra. Moreover,

$$\mathbb{F}(x_1, \ldots, x_k, x_{k+1}^p, \ldots, x_n^p) = \mathcal{C}_{\mathrm{D}}(\mathbb{F}(V))$$

for D generated by the Δ-relation of $\mathbb{H}$ and its translates $\mathbf{d}_i$ of length $\lambda_{\mathrm{H}} = k$ (see Corollary 3.9 and Proposition 3.10).

We prove the converse. Set $\lambda = \lambda_{\mathbb{H}}$. Let $\mathbf{d}$ be the Δ-relation of $\mathbb{H}$. Then

$$\mathbf{d} = \mathbf{d}_{\lambda,0}\mathscr{P}^{\Delta_0} + \cdots + (-1)^\lambda \mathbf{d}_{\lambda,\lambda}\mathscr{P}^{\Delta_\lambda}$$

vanishes on $\mathbb{H}$. Let $U \leq V$ be a vector subspace of dimension λ. We also note that the field $\mathbb{F}(U) \otimes \mathbb{F}(V/U)^p$ has Δ-relation $\mathbf{d}$ and Δ-length λ by Corollary 3.9. Certainly,

$$\mathbb{F}(V)^p \hookrightarrow \mathbb{F}(U) \otimes \mathbb{F}(V/U)^p \hookrightarrow \mathbb{F}(V)$$

is a purely inseparable extension of exponent one. We show that $\mathbb{F}(U) \otimes \mathbb{F}(V/U)^p \hookrightarrow \mathbb{H}$. Since $\mathbb{H} \hookrightarrow \mathbb{F}(V)$ is purely inseparable of exponent one, we have

$$\mathbb{F}(V/U)^p \hookrightarrow \mathbb{H}.$$

Since the coefficients of the Δ-relation are the Dickson classes, we know that $FF(\mathcal{D}(\lambda)) \hookrightarrow \mathbb{H}$. Thus

$$FF(\mathcal{D}(\lambda)) \otimes \mathbb{F}(V/U)^p \hookrightarrow \mathbb{H}.$$

Since $\mathbb{H} \hookrightarrow \mathbb{F}(V)$ is purely inseparable, we find that the separable closure of $FF(\mathcal{D}(\lambda)) \otimes \mathbb{F}[V/U]^p$ is in $\mathbb{H}$. This in turn is just

$$\mathbb{F}(U) \otimes \mathbb{F}(V/U)^p \hookrightarrow \mathbb{H}.$$

Obviously $|\mathbb{F}(V) \colon \mathbb{F}(U) \otimes \mathbb{F}(V/U)^p| = p^{n-\lambda}$. By Theorem 19 on page 186 of [1] we have that also

$$|\mathbb{F}(V) \colon \mathbb{H}| = p^{n-\lambda}$$

because $\mathcal{D}_{\mathbb{H}}$ has dimension $n - \lambda$ (Proposition 3.10). Hence $\mathbb{H} = \mathbb{F}(U) \otimes \mathbb{F}(V/U)^p$ as desired. □

Corollary 3.12. *Let* $\mathrm{H} \subseteq \mathbb{F}[V]$ *be a purely inseparable extension of exponent one. Let* H *be integrally closed. Then* H *is an unstable algebra over the Steenrod algebra if and only if* $\mathrm{H} = \mathbb{F}[x_1, \ldots, x_\lambda, x_{\lambda+1}^p, \ldots, x_n^p]$, *where* $\lambda = \lambda_{\mathrm{H}}$ *is the* Δ*-length of* H.

Proof. If H is an unstable algebra over the Steenrod algebra, then $\mathbb{H}$ is a field over the Steenrod algebra. Moreover, since $\mathrm{H} \hookrightarrow \mathbb{F}[V]$ has exponent one, so has the extension $\mathbb{H} \hookrightarrow \mathbb{F}(V)$. Thus $\mathbb{H} = \mathbb{F}(U) \otimes \mathbb{F}(V/U)^p$ for $\dim_{\mathbb{F}}(U) = \lambda$ by Theorem 3.11. Hence by Theorem 2.4 in [4]

$$\mathrm{H} = \overline{\mathrm{H}} = \mathcal{U}n(\mathbb{H}) = \mathbb{F}[U] \otimes \mathbb{F}[V/U]^p.$$

Conversely, the algebra $\mathbb{F}[U] \otimes \mathbb{F}[V/U]^p$ is certainly an unstable algebra over the Steenrod algebra. □

Remark. For any unstable integral domain H its integral closure $\overline{\mathrm{H}}$ also carries an unstable $\mathcal{P}^*$-module structure because $\overline{\mathrm{H}} = \mathcal{U}n(\mathbb{H})$ (see Theorem 2.4 in [4]). The converse is not true as we illustrate with the next example.

Example 3.13. Let $\mathbb{F}$ be the prime field of characteristic 2 and let A be the subalgebra of $\mathbb{F}[x, y]$ generated by x, xy, y^3. Then $\mathrm{A} \hookrightarrow \mathbb{F}[x, y]$ is an integral extension. Moreover, $FF(\mathrm{A}) = \mathbb{F}(x, y)$. Therefore $\overline{\mathrm{A}} = \mathbb{F}[x, y]$ is an unstable algebra over the Steenrod algebra. However A does not carry a $\mathcal{P}^*$-module structure because

$$\mathscr{P}^1(xy) = x^2y + xy^2 \notin \mathrm{A},$$

as the only elements of degree 3 in A are x^3, x^2, y^3.

Thus the assumption $\mathrm{H} = \overline{\mathrm{H}}$ cannot be dropped in the preceding result.

Corollary 3.14. *Let $U \leq V$. Denote $m = \dim_{\mathbb{F}}(U) \leq n = \dim_{\mathbb{F}}(V)$. Then*

$$\mathbb{F}[U] \otimes \mathbb{F}[V/U]^p \hookrightarrow \mathbb{F}[V]$$

is the largest unstable subalgebra with Δ-length equal to m.

Proof. Certainly, $\mathbb{F}[U] \otimes \mathbb{F}[V/U]^p$ has Δ-length m. Let

$$(\star) \qquad \mathbb{F}[U] \times \mathbb{F}[V/U]^p \hookrightarrow \mathrm{H} \hookrightarrow \mathbb{F}[V]$$

be an intermediate unstable algebra with $\lambda_{\mathrm{H}} = m$. Since the extension $(\star)$ is purely inseparable of exponent one, we have that

$$\mathbb{H} = \mathbb{F}(U') \otimes \mathbb{F}(V/U')^p$$

for some $U' \geq U$. But

$$\dim_{\mathbb{F}}(U) = m = \lambda_{\mathbb{H}} = \dim_{\mathbb{F}}(U')$$

and therefore $U = U'$. Hence

$$\mathbb{F}[U] \otimes \mathbb{F}[V/U]^p \subseteq \mathrm{H} \subseteq \mathcal{U}n(\mathrm{H}) = \overline{\mathrm{H}} = \mathbb{F}[U] \otimes \mathbb{F}[V/U]^p$$

gives the desired result. □

4. Purely inseparable extensions of arbitrary exponent

In this section we proceed with the investigation of the purely inseparable extension

$$\mathrm{H} \hookrightarrow \mathbb{F}[V].$$

We consider the general case of exponent $e \geq 1$. Thus we need to detect p^sth powers for $s = 1, \ldots, e$. We introduce the following operators for $s \in \mathbb{N}_0$:

$$\begin{aligned} \mathscr{P}^{\Delta_{s,0}} &= \frac{1}{p^s} \deg(-)\mathrm{id}(-), \\ \mathscr{P}^{\Delta_{s,1}} &= \mathscr{P}^{p^s}, \\ \mathscr{P}^{\Delta_{s,i}} &= \mathscr{P}^{p^s q^{i-1}} \mathscr{P}^{\Delta_{s,i-1}} - \mathscr{P}^{\Delta_{s,i-1}} \mathscr{P}^{p^s q^{i-1}} \quad \text{for } i \geq 2. \end{aligned}$$

Remark. Note that for all $s \in \mathbb{N}_0$ we have $\mathscr{P}^{\Delta_{s,i}} \in \mathcal{P}^*$ whenever $i \neq 0$.

Remark. Note also that the degree of $\mathscr{P}^{\Delta_{s,i}}$ is equal to $q^i p^s - p^s$, for all $i, s \in \mathbb{N}_0$.

Proposition 4.1. *The operators $\mathscr{P}^{\Delta_{s,i}}$ satisfy the following properties:*

(1) *For all $i \in \mathbb{N}$ we have $\mathscr{P}^{\Delta_{s,i}}(h^p) = (\mathscr{P}^{\Delta_{s-1,i}}(h))^p$ for $h \in \mathrm{H}$ and $s \geq 1$.*

(2) *For $i \geq 1$ and $k, s \geq 0$ we have*

$$[\mathscr{P}^{p^s k}, \mathscr{P}^{\Delta_{s,i}}] = \mathscr{P}^{\Delta_{s,i+1}} \mathscr{P}^{p^s k - p^s q^i}.$$

(3) *For $i, j \geq 1$ and $s \geq 0$ we have*

$$[\mathcal{P}^{\Delta_{s,i}}, \mathcal{P}^{\Delta_{s,j}}] = 0.$$

(4) *The pth iteration $\mathscr{P}^{\Delta_{s,i}} \cdots \mathscr{P}^{\Delta_{s,i}} = 0$ for all $i \geq 1$ and $s \geq 0$.*

Proof. AD(1): For any $i, j \geq 0$ and any linear form l we have

$$\mathscr{P}^i(l^j) = \binom{j}{i} l^{iq+j-i}$$

as it can be easily seen by induction. Thus for all $i \geq 0$ we have

$$\mathscr{P}^i(l^{p^s}) = \binom{p^s}{i} l^{iq+p^s-i}.$$

Since $\binom{p^s}{i} \equiv 0\,(p)$ precisely when $i \notin \{p^s, 0\}$, we have

$$\mathscr{P}^i(l^{p^s}) = \begin{cases} \mathscr{P}^0(l)p^s & \text{for } i = 0, \\ \mathscr{P}^1(l)^{p^s} & \text{for } i = p^s, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\mathscr{P}^i(h^{p^s}) = \begin{cases} h^{p^s} & \text{for } i = 0, \\ \mathscr{P}^k(h)^{p^s} & \text{for } i = kp^s, \\ 0 & \text{otherwise,} \end{cases}$$

for any $h \in \mathrm{H}$ (cf. page 261 of [6]). Thus

$$\mathscr{P}^{\Delta_{s,1}}(h^p) = \mathscr{P}^{p^s}(h^p) = (\mathscr{P}^{p^{s-1}}(h))^p = (\mathscr{P}^{\Delta_{s-1,1}}(h))^p.$$

Thus by induction on i we find

$$\begin{aligned} \mathscr{P}^{\Delta_{s,i}}(h^p) &= \mathscr{P}^{p^s q^{i-1}} \mathscr{P}^{\Delta_{s,i-1}}(h^p) - \mathscr{P}^{\Delta_{s,i-1}} \mathscr{P}^{p^s q^{i-1}}(h^p) \\ &= \mathscr{P}^{p^s q^{i-1}} (\mathscr{P}^{\Delta_{s-1,i-1}}(h))^p - \mathscr{P}^{\Delta_{s,i-1}} (\mathscr{P}^{p^{s-1} q^{i-1}}(h))^p \\ &= (\mathscr{P}^{p^{s-1} q^{i-1}} \mathscr{P}^{\Delta_{s-1,i-1}}(h) - \mathscr{P}^{\Delta_{s-1,i-1}} \mathscr{P}^{p^{s-1} q^{i-1}}(h))^p \\ &= (\mathscr{P}^{\Delta_{s-1,i}}(h))^p, \end{aligned}$$

as claimed.

AD(2) and (3): The result follows, because it is true for any linear form.

AD(4): From the Adem relations it follows that

$$\mathscr{P}^{p^s} \cdots \mathscr{P}^{p^s} = 0.$$

Thus the result follows by induction on i with the help of the commutation rules of (2) (cf. Lemma A.1.1 in [3]). □

Define the H^{p^s}-module

$$\mathrm{Der}_{\mathrm{H},s} = \mathrm{span}_{\mathrm{H}^{p^s}}\{\mathscr{P}^{\Delta_{s,i}} \mid i \in \mathbb{N}_0\}.$$

By definition it follows that $\mathrm{Der}_{\mathrm{H},0} = \mathrm{Der}_{\mathrm{H}}$.

Proposition 4.2. *The module* $\mathrm{Der}_{\mathrm{H},s}$ *has the following properties:*

(1) $\mathrm{Der}_{\mathrm{H},s}$ *acts in* H^{p^s} *as derivations.*

(2) *For* $s, k \geq 0$ *we obtain*

$$[\mathscr{P}^{p^s k}, \mathscr{P}^{\Delta_{s,0}}] = k\mathscr{P}^{kp^s}.$$

(3) *If* $s \geq 0$*, then*

$$[\mathscr{P}^{\Delta_{s,i}}, \mathscr{P}^{\Delta_{s,j}}] = \begin{cases} \mathscr{P}^{\Delta_{s,i}} & \text{if } i \neq 0 \text{ and } j = 0, \\ -\mathscr{P}^{\Delta_{s,j}} & \text{if } i = 0 \text{ and } j \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(4) *The pth iteration gives* $\mathscr{P}^{\Delta_{s,0}} \cdots \mathscr{P}^{\Delta_{s,0}}(h^{p^s}) = \mathscr{P}^{\Delta_{s,0}}(h^{p^s})$ *for all* $s \geq 0$ *and* $h \in H$.

(5) *Let* $s \geq 0$*. Then* $\mathscr{P}^{\Delta_{s,i}}(h^{p^s}) = 0 \ \forall i$ *if and only if* h *is a* p^{s+1}*st power.*

Proof. AD(1): Let $h^{p^s} \in \mathrm{H}^{p^s}$. By Proposition 4.1 $\mathrm{Der}_{\mathrm{H},s}$ acts on H^{p^s} according to the following formulae:

$$\mathscr{P}^{\Delta_{s,0}}(h^{p^s}) = \deg(h)h^{p^s} = \mathscr{P}^{\Delta_0}(h)^{p^s},$$
$$\mathscr{P}^{\Delta_{s,1}}(h^{p^s}) = (\mathscr{P}^1(h))^{p^s} = (\mathscr{P}^{\Delta_1}(h))^{p^s},$$
$$\mathscr{P}^{\Delta_{s,i}}(h^{p^s}) = (\mathscr{P}^{q^{i-1}}\mathscr{P}^{\Delta_{i-1}} - \mathscr{P}^{\Delta_{i-1}}\mathscr{P}^{q^{i-1}}(h))^{p^s} = (\mathscr{P}^{\Delta_i}(h))^{p^s}.$$

Since taking pth powers is additive in characteristic p, this establishes the statement.

AD(2): Let $h^{p^s} \in \mathrm{H}^{p^s}$. We have

$$\begin{aligned}[\mathscr{P}^{p^s k}, \mathscr{P}^{\Delta_{s,0}}](h^{p^s}) &= \mathscr{P}^{p^s k}\mathscr{P}^{\Delta_{s,0}}(h^{p^s}) - \mathscr{P}^{\Delta_{s,0}}\mathscr{P}^{p^s k}(h^{p^s}) \\ &= \deg(h)\mathscr{P}^{p^s k}(h^{p^s}) - \mathscr{P}^{\Delta_{s,0}}(\mathscr{P}^k(h))^{p^s} \\ &= \deg(h)\mathscr{P}^{p^s k}(h^{p^s}) - \deg(\mathscr{P}^k(h))\mathscr{P}^{p^s k}(h^{p^s}) \\ &= (\deg(h) - \deg(h) + k - kq)\mathscr{P}^{p^s k}(h^{p^s}) \\ &= k\mathscr{P}^{p^s k}(h^{p^s}).\end{aligned}$$

AD(3): Let $h^{p^s} \in \mathrm{H}^{p^s}$. If $i, j \geq 1$, then

$$[\mathscr{P}^{\Delta_{s,0}}, \mathscr{P}^{\Delta_{s,j}}] = 0$$

by part (3) of Proposition 4.1. Otherwise we have

$$\begin{aligned}[\mathscr{P}^{\Delta_{s,0}}, \mathscr{P}^{\Delta_{s,j}}](h^{p^s}) &= \mathscr{P}^{\Delta_{s,0}}\mathscr{P}^{\Delta_{s,j}}(h^{p^s}) - \mathscr{P}^{\Delta_{s,j}}\mathscr{P}^{\Delta_{s,0}}(h^{p^s}) \\ &= (\deg(h) + q^j - 1)\mathscr{P}^{\Delta_{s,j}}(h^{p^s}) - \deg(h)\mathscr{P}^{\Delta_{s,j}}(h^{p^s}) \\ &= -\mathscr{P}^{\Delta_{s,j}}(h^{p^s}).\end{aligned}$$

The relation for $j = 0$ can be established in the same way.

AD(4): Let $h^{p^s} \in \mathrm{H}^{p^s}$. Since $\mathscr{P}^{\Delta_{s0}}(h^{p^s}) = \deg(h)h^{p^s}$ and $\deg(h)^{p^s} = \deg(h)$ the result follows.

Ad(5): Let $h^{p^s} = \mathrm{H}^{p^s}$ and $i \geq 1$. Then

$$\mathscr{P}^{\Delta_{s,i}}(h^{p^s}) = (\mathscr{P}^{\Delta_i}(h))^{p^s}$$

and

$$\mathscr{P}^{\Delta_{s,0}}(h^{p^s}) = \deg(h)h^{p^s} = \mathscr{P}^{\Delta_0}(h)^{p^s}$$

are simultaneously zero for all i if and only if

$$\mathscr{P}^{\Delta_i}(h) = 0$$

for all $i \geq 0$, i.e., if and only if h is a pth power; hence precisely when h^{p^s} is a p^{s+1}st power. □

Thus $\mathrm{Der}_{\mathrm{H},s}$ is a restricted Lie algebra of derivations acting on H^{p^s} vanishing precisely on the p^{s+1}st powers. We need to have a look at the relation between $\mathrm{Der}_{\mathrm{H},s}$ and $\mathrm{Der}_{\mathrm{H}}$.

Proposition 4.3. *Let* H *be an unstable reduced Noetherian algebra over the Steenrod algebra. The action of* $\mathrm{Der}_{\mathrm{H},s}$ *on* H^{p^s} *has the following properties:*

(1) *For any* $d_s \in \mathrm{Der}_{\mathrm{H},s}$ *there exists a* $d \in \mathrm{Der}_{\mathrm{H}}$ *such that*

$$d_s(h^{p^s}) = d(h)^{p^s} \quad \forall h \in \mathrm{H}.$$

(2) *If there are* m *derivations in* $\mathrm{Der}_{\mathrm{H},s}$ *that are linearly dependent, then so are any* m *derivations.*

Proof. AD(1): For any

$$d_s = h_0^{p^s}\mathscr{P}^{\Delta_{s,i_0}} + \cdots + h_l^{p^s}\mathscr{P}^{\Delta_{s,i_l}} \in \mathrm{Der}_{\mathrm{H},s}$$

we find that

$$d_s = (d)^{p^s} = (h_0\mathscr{P}^{\Delta_{i_0}} + \cdots + h_l\mathscr{P}^{\Delta_{i_l}})^{p^s}$$

by part (1) of Proposition 4.1. By construction $d \in \mathrm{Der}_{\mathrm{H}}$.

AD(2): Let $d_{s,1}, \ldots, d_{s,m} \in \mathrm{Der}_{\mathrm{H},s}$ be linearly dependent. By part (1) we find $d_1, \ldots, d_m \in \mathrm{Der}_{\mathrm{H}}$ such that $(d_i(h))^{p^s} = d_{s,i}(h^{p^s})$ for all $h \in \mathrm{H}$. Thus the elements $d_1, \ldots, d_m \in \mathrm{Der}_{\mathrm{H}}$ are linearly dependent. Therefore any m elements, say $d'_1, \ldots, d'_m$, of $\mathrm{Der}_{\mathrm{H}}$ are linearly dependent with a relation

$$h_1 d'_1 \cdots + h_m d'_m = 0.$$

Thus

$$h_1^{p^s} d'_{s,1} + \cdots + h_m^{p^s} d'_{s,m} = 0$$

is a relation in $\mathrm{Der}_{\mathrm{H},s}$. □

Thus the minimal l_s such that $l_s + 1$ elements of $\mathrm{Der}_{\mathrm{H},s}$ are linearly dependent is uniquely defined. We call l_s the Δ_s-*length* of H^{p^s}, denoted by $\lambda_{\mathrm{H},s}$ or if no confusion can arise by λ_s. If $\lambda_s \in \mathbb{N}_0$, we call the algebra H^{p^s} Δ_s-*finite*. Note that by construction H is Δ-finite if and only if H^{p^s} is Δ_s-finite.

Proposition 4.4. *Let* H *be an unstable reduced* Δ*-finite algebra over the Steenrod algebra. Let* λ *be its* Δ*-length and* λ_s *the* Δ_s*-length of* H^{p^s}*. Then*

(1) $\lambda_s = \lambda$.

(2) *We have a relation of the form*

$$\mathbf{d}_s = \mathbf{d}_{\lambda_s,0}^{p^s}\mathscr{P}^{\Delta_{s,0}} + \cdots + \mathbf{d}_{\lambda_s,\lambda_s}^{p^s}\mathscr{P}^{\Delta_{s,\lambda_s}}$$

on H^{p^s}.

Proof. AD(1): Let

$$d = \mathbf{d}_{\lambda,0}\mathscr{P}^{\Delta_0} + \cdots + (-1)^{\lambda}\mathbf{d}_{\lambda,\lambda}\mathscr{P}^{\Delta_\lambda} = \mathrm{Der}_{\mathrm{H}}$$

be the Δ-relation of H. Then the element $d_s(h^{p^s}) = (d(h))^{p^s}$ of $\mathrm{Der}_{\mathrm{H},s}$ vanishes on H^{p^s}. Thus $\lambda_s \leq \lambda$. Conversely, if

$$d_s = h_0^{p^s} \mathscr{P}^{\Delta_{s,0}} + \cdots + h_{\lambda_s}^{p^s} \mathscr{P}^{\Delta_{s,\lambda_s}} \in \mathrm{Der}_{\mathrm{H},s}$$

is a Δ_s-relation of H^{p^s}, then the element

$$d = h_0 \mathscr{P}^{\Delta_0} + \cdots + h_{\lambda_s} \mathscr{P}^{\Delta_{\lambda_s}} \in \mathrm{Der}_{\mathrm{H}}$$

vanishes on H, by part (1) of Proposition 4.3. Thus, also, $\lambda \leq \lambda_s$.

AD(2): By (1) λ_s is the Δ-length of H. Therefore

$$\mathbf{d} = \mathbf{d}_{\lambda_s,0} \mathscr{P}^{\Delta_0} + \cdots + \mathbf{d}_{\lambda_s,\lambda_s} \mathscr{P}^{\Delta_{\lambda_s}}$$

is the Δ-relation on H. Thus the result follows by part (1) of Proposition 4.3. □

We call the relation $\mathbf{d}_s$ of part (2) of this result the Δ_s-*relation* of H^{p_s}.

Let $\mathrm{H}^{p^s} \hookrightarrow \mathbb{F}[V]^{p^s}$ with Δ_s-length λ_s. Then $\mathrm{H} = \mathrm{H}_s^{p^s} \hookrightarrow \mathbb{F}[V]$, since $\mathrm{H}_s^{p^s}$ arises from H^{p^s} by adjoining all p^sth roots. Let H have Δ-length λ. Thus $\lambda = \lambda_s \leq n$ by Proposition 4.4, part (1).

Proposition 4.5. *For* $s \geq 0$ *we have*

$$(\mathcal{C}_{\mathrm{Der}_{\mathrm{H}}}(\mathrm{H}))^{p^s} = \mathcal{C}_{\mathrm{Der}_{\mathrm{H},s}}(\mathrm{H}^{p^s}) \subseteq \mathcal{C}_{\mathrm{Der}_{\mathrm{H}}}(\mathrm{H}).$$

Proof. If $h^{p^s} \in \mathrm{H}^{p^s}$ is a $\mathrm{Der}_{\mathrm{H},s}$-constant, then h is a p^{s+1}st power by part (5) of Proposition 4.2. Thus there exists an element $k \in \mathrm{H}_1^{p^s} = \mathrm{H}^{p^{s-1}}$ such that $k^p = h^{p^s}$. Hence $h^{p^s} = k^p \in \mathrm{H}_1^{p^s} \subseteq \mathrm{H}$ is a pth power, i.e., a $\mathrm{Der}_{\mathrm{H}}$-constant.

In order to prove the equality, let $h \in \mathrm{H}$ be a $\mathrm{Der}_{\mathrm{H}}$-constant, i.e., $d(h) = 0$ for all $d \in \mathrm{Der}_{\mathrm{H}}$. For any $d_s \in \mathrm{Der}_{\mathrm{H},s}$ there is an element $d \in \mathrm{Der}_{\mathrm{H}}$ such that $d_s(h^{p^s}) = (d(h))^{p^s} = 0$ by part (1) of Proposition 4.3. Thus $(\mathcal{C}_{\mathrm{Der}_{\mathrm{H}}}(\mathrm{H}))^{p^s} \subseteq \mathcal{C}_{\mathrm{Der}_{\mathrm{H},s}}(\mathrm{H}^{p^s})$. Conversely, if $h^{p^s} \in \mathcal{C}_{\mathrm{Der}_{\mathrm{H},s}}(\mathrm{H}^{p^s})$, then $h \in \mathcal{C}_{\mathrm{Der}_{\mathrm{H}}}(\mathrm{H})$, by what we have proven so far. Thus $h^{p^s} \in (\mathcal{C}_{\mathrm{Der}_{\mathrm{H}}}(\mathrm{H}))^{p^s}$ as desired. □

Remark. We note that the preceding results imply that

$$\dim(\mathrm{Der}_{\mathbb{F}[V],s}) = \dim(\mathrm{Der}_{\mathbb{F}[V]}) = \dim_{\mathbb{F}}(V) = n.$$

Remark. In Section 3 we defined $\mathrm{Der}_{\mathbb{H}}$ for $\mathbb{H} = FF(\mathrm{H})$ for integral domains H. In the same way we define

$$\mathrm{Der}_{\mathbb{H},s} = \mathrm{span}_{\mathbb{H}^{p^s}}\{\mathscr{P}^{\Delta_{i,s}} \mid i \in \mathbb{N}_0\}.$$

We obtain analogously to Proposition 3.5 that $\dim(\mathrm{Der}_{\mathbb{H},s}) = \dim(\mathrm{Der}_{\mathrm{H},s})$. Hence, $\lambda_{\mathbb{H},s} = \lambda_{\mathrm{H},s}$ and $\mathbf{d}_{\mathbb{H},s} = \mathbf{d}_{\mathrm{H},s}$. Therefore $\mathrm{Der}_{\mathbb{H},s}$ and $\mathrm{Der}_{\mathrm{H},s}$ have basis $\{\mathscr{P}^{\Delta_{s,0}}, \ldots, \mathscr{P}^{\Delta_{s,\lambda_s}}\}$, where $\lambda_s = \lambda_{\mathbb{H},s} = \lambda_{\mathrm{H},s}$.

Corollary 4.6. *The* Δ_s*-length of* $\mathbb{F}[V]^{p^s}$*, resp.* $\mathbb{F}(V)^{p^s}$*, is equal to the dimension of* V*. Moreover, for* $\mathrm{D}_s = 0$

$$\mathbb{F}[V]^{p^s} = \mathcal{C}_{\mathrm{D}_s}(\mathbb{F}[V]^{p^s}), \quad \textit{resp. } \mathbb{F}(V)^{p^s} = \mathcal{C}_{\mathrm{D}_s}(\mathbb{F}(V)^{p^s}).$$

Finally, the Δ_s*-length of* $\mathbb{F}[V]^{p^{s+1}}$*, resp.* $\mathbb{F}(V)^{p^{s+1}}$*, is zero, and*

$$\mathbb{F}[V]^{p^{s+1}} = \mathcal{C}_{\mathrm{Der}_{\mathbb{F}[V],s}}(\mathbb{F}[V]^{p^s}), \quad \textit{resp. } \mathbb{F}(V)^{p^{s+1}} = \mathcal{C}_{\mathrm{Der}_{\mathbb{F}(V),s}}(\mathbb{F}(V)^{p^s}).$$

Proof. The first statement follows from Lemma 3.8 and Proposition 4.5. The second statement follows from Corollaries 3.2 and 3.9 and Proposition 4.5. □

We need a generalization of Lemma 3.7.

Lemma 4.7. *Let* $\mathrm{K}^{p^s} \subseteq \mathrm{H}^{p^s}$ *be unstable reduced Noetherian algebras over the Steenrod algebra. Denote by* $\lambda_{\mathrm{K},s}$*, resp.* $\lambda_{\mathrm{H},s}$*, the* Δ_s*-length of* K^{p^s}*, resp.* H^{p^s}*. Then* $\lambda_{\mathrm{K},s} \leq \lambda_{\mathrm{H},s}$*.*

Proof. The proof works just as the one of Lemma 3.7. □

Proposition 4.8. *Let* $t > s$ *and* $W \leq V$*. The* Δ_s*-length of* $\mathbb{F}[W]^{p^s} \otimes \mathbb{F}[V/W]^{p^t}$ *as well as of* $\mathbb{F}(W)^{p^s} \otimes \mathbb{F}(V/W)^{p^t}$*, is equal to* $\lambda_s = \dim_{\mathbb{F}}(W)$*. Moreover, let* $\mathrm{H}^{p^s} \hookrightarrow \mathbb{F}[V]^{p^s}$*. Then the* Δ_s*-length of* H^{p^s} *is at most* l *if and only if*

$$\mathrm{H}^{p^s} \subseteq \mathbb{F}[W]^{p^s} \otimes \mathbb{F}[V/W]^{p^t},$$

where $\dim_{\mathbb{F}}(W) = l$ *and* $t > s$*.*

Proof. The first statement follows from Proposition 4.4, Lemma 3.8, and Corollary 3.9. If $\mathrm{H}^{p^s} \subseteq \mathbb{F}[W]^{p^s} \otimes \mathbb{F}[V/W]^{p^t}$, then by Lemma 4.7 its Δ_s-length is at most the Δ_s-length of $\mathbb{F}[W]^{p^s} \otimes \mathbb{F}[V/W]^{p^t}$, which is l by what we proved so far.

Conversely, let the Δ_s-length of H^{p^s} be $\lambda_s \leq l$. Then H has Δ-length $\lambda = \lambda_s$ by Proposition 4.4. Thus its Δ-relation is

$$\mathbf{d} = \mathbf{d}_{\lambda,0}\mathscr{P}^{\Delta_0} + \cdots \mathbf{d}_{\lambda,\lambda}\mathscr{P}^{\Delta_\lambda}$$

and therefore the Dickson algebra of dimension λ is contained in H.

$$\mathcal{D}(\lambda) \hookrightarrow \mathrm{H} \hookrightarrow \mathbb{F}[U] \otimes \mathbb{F}[V/U]$$

for some $U \leq V$ with $\dim(U) = \lambda$. But $\mathbb{F}[U] \otimes \mathbb{F}[V/U]^p$ is the largest subalgebra of $\mathbb{F}[V]$ with Δ-length λ by Corollary 3.14. Thus

$$\mathrm{H} \hookrightarrow \mathbb{F}[U] \otimes \mathbb{F}[V/U]^p \hookrightarrow \mathbb{F}[W] \otimes \mathbb{F}[V/W]^p$$

and the result follows. □

Lemma 4.9. *We have*

$$FF(\mathcal{C}_{\mathrm{Der}_{\mathrm{H},s}}(\mathrm{H}^{p^s})) = \mathcal{C}_{\mathrm{Der}_{\mathbb{H},s}}(\mathbb{H}^{p^s}).$$

Proof. This follows from Propositions 4.5 and 2.3. □

In order to be able to treat the general case, we need another preliminary result.

Proposition 4.10. *Let* $V = W_0 \oplus \cdots \oplus W_e$ *be a vector space decomposition. Consider the purely inseparable extension* $\mathrm{H}^{p^{s_0}} \hookrightarrow \mathbb{F}[W_0]^{p^{s_0}} \otimes \cdots \otimes \mathbb{F}[W_e]^{p^{s_e}}$*. Let* $s_0 < \cdots < s_e$*. Let* $\mathrm{H}^{p^{s_0}}$ *be an integrally closed unstable algebra over the Steenrod algebra. Then the* Δ_{s_0}*-length of* $\mathrm{H}^{p^{s_0}}$ *is* λ_{s_0} *if and only if*

$$\mathbb{F}[U_0]^{p^{s_0}} \hookrightarrow \mathbb{H}^{p^{s_0}} \hookrightarrow \mathbb{F}[U-0]^{p^{s_0}} \otimes \mathbb{F}[W_0/U_0]^{p^{s_0+1}} \otimes \mathbb{F}[W_1]^{p^{s_1}} \otimes \cdots \otimes \mathbb{F}[W_e]^{p^{s_e}}$$

for $\dim_{\mathbb{F}}(U_0) = \lambda_{s_0}$*,* $U_0 \leq W_0$ *and* U_0 *maximal with respect to this property.*

Proof. Let the Δ_{s_0}-length of $\mathrm{H}^{p_{s_0}}$ be λ_{s_0}. Note that $\lambda_{s_0} \leq \dim_{\mathbb{F}}(W_0)$ by Lemma 4.7. The Δ-length of H is also λ_{s_0} by part (1) of Proposition 4.4. Moreover

$$\mathrm{H} \hookrightarrow \mathbb{F}[W_0] \otimes \mathbb{F}[W_1]^{p^{s_1-s_0}} \otimes \cdots \otimes \mathbb{F}[W_e]^{p^{s_e-s_0}}.$$

Let **d** be the Δ-relation of H. Then **d** is also the Δ-relation of its field of fractions, $\mathbb{H}$. Thus by construction we find for $\mathrm{D} = \mathrm{D}_{\mathbb{H}} \subseteq \mathrm{Der}_{\mathbb{H}}$

$$\begin{aligned}\mathbb{H} \hookrightarrow \mathcal{C}_{\mathrm{D}}(\mathbb{F}(W_0) \otimes \mathbb{F}(W_1)^{p^{s_1 - s_0}} \otimes \cdots \otimes \mathbb{F}(W_e)^{p^{s_e - s_0}}) \\ = \mathbb{F}(U_0) \otimes \mathbb{F}(W_0/U_0)^p \otimes \mathbb{F}(W_1)^{p^{s_1-s_0}} \otimes \cdots \otimes \mathbb{F}(W_e)^{p^{s_e - s_0}}\end{aligned}$$

for some $U_0 \leq W_0$ of dimension λ_{s_0} by Corollary 3.9. Let U be maximal with this property. The extension

$$\mathbb{F}(U_0) \cap \mathbb{H} \hookrightarrow \mathbb{F}(U_0)$$

is purely inseparable, since

$$\mathbb{H} \hookrightarrow \mathbb{F}(U_0) \otimes \mathbb{F}(W_0/U_0)^p \otimes \mathbb{F}(W_1)^{p^{s_1-s_0}} \otimes \cdots \otimes \mathbb{F}(W_e)^{p^{s_e-s_0}}$$

is purely inseparable and all elements in $\mathbb{F}(U_0) \otimes \mathbb{F}(W_0/U_0)^p \otimes \mathbb{F}(W_1)^{p^{s_1-s_0}} \otimes \cdots \otimes \mathbb{F}(W_e)^{p^{s_e-s_0}}$ that are algebraic over $\mathbb{F}(U_0)$ are inseparable. By maximality of λ_{s_0} all elements in $\mathbb{F}(U_0)$ are separable over $\mathbb{H}$. Thus

$$\mathbb{F}(U_0) \cap \mathbb{H} \hookrightarrow \mathbb{F}(U_0)$$

is also separable. Hence

$$\mathbb{F}(U_0) = \mathbb{F}(U_0) \cap \mathbb{H} \hookrightarrow \mathbb{H}.$$

Therefore

$$\mathbb{F}[U_0]^{p^{s_0}} \hookrightarrow \mathrm{H}^{p^{s_0}} \hookrightarrow \mathbb{F}[U_0]^{p^{s_0}} \otimes \mathbb{F}[W_0/U_0]^{p^{s_0+1}} \otimes \cdots \otimes \mathbb{F}[W_e]^{p^{s_e}}$$

as desired.

To prove the converse, assume that there exists a vector space $U_0 \leq W_0$ of $\dim_{\mathbb{F}}(U_0) = \lambda_{s_0}$ such that

$$\mathbb{F}[U_0]^{p^{s_0}} \hookrightarrow \mathrm{H}^{p^{s_0}} \hookrightarrow \mathbb{F}[U_0]^{p^{s_0}} \otimes \mathbb{F}[W_0/U_0]^{p^{s_0+1}} \otimes \cdots \otimes \mathbb{F}[W_e]^{p^{s_e}}.$$

Assume furthermore that U_0 is maximal with this property. Then the Δ_{s_0}-length of $\mathbb{F}[U_0]^{p^{s_0}}$ is $\lambda_{s,0}$ by Corollary 4.6. Equally, the Δ_{s_0}-length of $\mathbb{F}[U_0]^{p^{s_0}} \otimes \mathbb{F}[W_0/U_0]^{p^{s_0+1}} \otimes \cdots \otimes \mathbb{F}[W_e]^{p^{s_e}}$ is λ_{s_0} by Proposition 4.8. Therefore the Δ_{s_0}-length of $H^{p^{s_0}}$ is λ_{s_0} by Lemma 4.7. $\square$

Remark. Note that any element in $\mathrm{H}^{p^{s_0}}$ that is algebraic over $\mathbb{F}[U_0]^{p^{s_0}}$ is separable over $\mathbb{F}[U_0]^{p^{s_0}}$.

Theorem 4.11. *Let $V = W_0 \oplus \cdots \oplus W_e$ be a vector space decomposition. Consider the purely inseparable extension $\mathrm{H} \hookrightarrow \mathbb{F}[W_0] \otimes \mathbb{F}[W_1]^p \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e}$ of exponent one. Let* H *be integrally closed. Then* H *is an unstable algebra over the Steenrod algebra if and only if*

$$\mathrm{H} = \mathbb{F}[U_0] \otimes \mathbb{F}[U_1]^p \otimes \cdots \otimes \mathbb{F}[U_{e+1}]^{p^{e+1}}$$

for some vector space decomposition

$$V = U_0 \oplus \cdots \oplus U_{e+1}$$

with

$$U_0 \oplus \cdots \oplus U_i \leq W_0 \oplus \cdots \oplus W_i$$

and $\dim(U_0 \oplus \cdots \oplus U_i)$ *is the* Δ*-length of* H_i*,* $i = 0, \ldots, e+1$*.*

Proof. The "if" part of the statement is clear by Proposition 4.10. We need to prove the "only if" part.

Let $\mathbf{d} \in \mathrm{Der}_{\mathrm{H}}$ be the Δ-relation on H of length λ_0. Then $\mathbb{F}[U_0] \hookrightarrow \mathrm{H}$ for some vector space U_0 of dimension λ_0 by Proposition 4.10. Hence $\lambda_0 \leq \dim(W_0)$, $U_0 \leq W_0$, and we have

$$\mathbb{F}[U_0] \hookrightarrow \mathrm{H} \hookrightarrow \mathbb{F}[W_0] \otimes \mathbb{F}[W_1]^p \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e}.$$

We consider the chain

$$\begin{aligned}\mathbb{F}[U_0]_1 \hookrightarrow \mathrm{H}_1 \hookrightarrow (\mathbb{F}[W_0] \otimes \mathbb{F}[W_1]^p \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e})_1 \\ = \mathbb{F}[W_0 \oplus W_1] \otimes \mathbb{F}[W_2]^p \otimes \cdots \otimes \mathbb{F}[W_e]^{p^{e-1}}.\end{aligned}$$

By Proposition 4.10 the Δ-length of H_1 is at most the dimension of $W_0 \oplus W_1$ and

$$\mathbb{F}[U_0] \otimes \mathbb{F}[U_1] \hookrightarrow \mathbb{H}_1$$

for a suitable $U_0 \oplus U_1 \leq W_0 \oplus W_1$. Since $\mathbb{F}[U_0] \hookrightarrow \mathrm{H}_1$, and U_0 is the maximal vector subspace with this property, we have

$$\mathbb{F}[U_0] \otimes \mathbb{F}[U_1]^p \hookrightarrow \mathrm{H}.$$

Proceeding inductively gives an extension

$$\mathbb{F}[U_0] \otimes \mathbb{F}[U_1]^p \otimes \cdots \otimes \mathbb{F}[U_{e+1}]^{p^{e+1}} \hookrightarrow \mathrm{H},$$

which is separable, because it is algebraic (cf. the remark after Proposition 4.10). This extension is also purely inseparable because

$$\mathbb{F}[U_0] \otimes \mathbb{F}[U_1]^p \otimes \cdots \otimes \mathbb{F}[U_{e+1}]^{p^{e+1}} \hookrightarrow \mathbb{F}[W_0] \otimes \mathbb{F}[W_1]^p \otimes \cdots \otimes \mathbb{F}[W_{e+1}]^{p^{e+1}}$$

is purely inseparable. Thus

$$\mathrm{H} = \mathbb{F}[U_0] \otimes \mathbb{F}[U_1]^p \otimes \cdots \otimes \mathbb{F}[U_{e+1}]^{p^{e+1}}$$

as desired. □

Corollary 4.12. *Let $V = W_0 \oplus \cdots \oplus W_e$ be a vector space decomposition. Let $\mathbb{H} \hookrightarrow \mathbb{F}(W_0) \otimes \mathbb{F}(W_1)^p \otimes \cdots \otimes \mathbb{F}(W_e)^{p^e}$ be a purely inseparable extension of exponent one. Then $\mathbb{H}$ is a field over the Steenrod algebra if and only if*

$$\mathbb{H} = \mathbb{F}(U_0) \otimes \mathbb{F}(U_1)^p \otimes \cdots \otimes \mathbb{F}(U_{e+1})^{p^{e+1}}$$

for some vector space decomposition

$$V = U_0 \oplus \cdots \oplus U_{e+1}$$

with

$$U_0 \oplus \cdots \oplus U_i \leq W_0 \oplus \cdots \oplus W_i$$

and $\dim(U_0 \oplus \cdots \oplus U_i)$ *is the* Δ*-length of* $\mathbb{H}_i$, $i = 0, \ldots, e+1$.

Proof. Since

$$\mathcal{U}n(\mathbb{H}) \hookrightarrow \mathbb{F}[U_0] \otimes \mathbb{F}[U_1]^p \otimes \cdots \otimes \mathbb{F}[U_e]^{p^e}$$

is integrally closed and $FF(\mathcal{U}n(\mathbb{H})) = \mathbb{H}$, the result follows from Theorem 4.11. □

Theorem 4.13. $\mathbb{H} \hookrightarrow \mathbb{F}(V)$ *is a purely inseparable extension of exponent* e *of fields over the Steenrod algebra if and only if*

$$\mathbb{H} = \mathbb{F}(W_0) \otimes \mathbb{F}(W_1)^p \otimes \cdots \otimes \mathbb{F}(W_e)^{p^e}$$

for some vector space decomposition

$$V = W_0 \oplus \cdots \oplus W_e,$$

where $\dim(W_0 \oplus \cdots \oplus W_i)$ *is the* Δ*-length of* $\mathbb{H}_i$, $i = 0, \ldots, e$.

Proof. The "if" part is clear by Corollary 4.12. We show the "only if" part.

We proceed by induction on e. The case $e = 1$ has been treated in Theorem 4.11. Thus assume that $e > 1$.

We have a chain of purely inseparable extensions of exponent one

$$\mathbb{H} = \mathbb{H}_0 \hookrightarrow \mathbb{H}_1 \hookrightarrow \cdots \hookrightarrow \mathbb{H}_e = \mathbb{F}(V)$$

which is obtained by adjoining successively pth roots. Note that all $\mathbb{H}_i$'s are fields over the Steenrod algebra.

By the induction hypothesis we can assume that

$$\mathbb{H} \hookrightarrow \mathbb{H}_1 = \mathbb{F}(W_0) \otimes \mathbb{F}(W_1)^p \otimes \cdots \otimes \mathbb{F}(W_{e-1})^{p^{e-1}}$$

for a vector space decomposition

$$V = W_0 \oplus \cdots \oplus W_{e-1}.$$

By Corollary 4.12 we are done. □

At the level of algebras we obtain the following result as an obvious corollary.

Corollary 4.14. *Let* H *be integrally closed. Let* $\mathrm{H} \hookrightarrow \mathbb{F}[V]$ *be a purely inseparable extension of exponent* e*. Then* H *is an algebra over the Steenrod algebra if and only if*

$$\mathrm{H} = \mathbb{F}[W_0] \otimes \mathbb{F}[W_1]^p \otimes \cdots \otimes \mathbb{F}[W_e^{p^e}]$$

for some vector space decomposition

$$V = W_0 \oplus \cdots \oplus W_e,$$

where $\dim(W_0 \oplus \cdots \oplus W_i)$ *is the* Δ*-length of* H_i, $i = 0, \ldots, e$. □

Remark. Note that Corollary 4.14 has been proven in Theorem 7.2.2 of [3] as well as in [7], Theorem II, without, however, the precise statement on the dimension of $W_0 \oplus \cdots \oplus W_i$.

5. Purely inseparable extensions, the general case

Let H be an unstable Noetherian integral domain over the Steenrod algebra. Assume that the canonical inclusion

$$\mathrm{H} \hookrightarrow \sqrt[\mathcal{P}^*]{\mathrm{H}}$$

is purely inseparable of exponent e.

If H is integrally closed, then so is $\sqrt[\mathcal{P}^*]{\mathrm{H}}$ by part (3) of Proposition 2.1. Then

$$\sqrt[\mathcal{P}^*]{\mathrm{H}} \hookrightarrow \mathbb{F}[V]$$

is a Galois extension with Galois group $G \leq \mathrm{GL}(n, \mathbb{F})$, where n is the Krull dimension of H (see the Galois Embedding Theorem, Theorem 7.1.1 in [3]). Thus

$$\sqrt[\mathcal{P}^*]{\mathrm{H}} = \mathbb{F}[V]^G.$$

On the other hand we can take the separable closure first: The separable closure of $\mathrm{H} \hookrightarrow \mathbb{F}[V]$ denoted by $\overline{\mathrm{H}}^{sep}$ is again an unstable algebra over the Steenrod algebra by the Separable Extension Lemma (Proposition 2.2.2 in [3]), since $\overline{\mathrm{H}}^{sep} = \mathcal{U}n(\overline{\mathbb{H}}^{sep})$. Thus we obtain a purely inseparable extension of exponent e

$$\overline{\mathrm{H}}^{sep} \hookrightarrow \mathbb{F}[V].$$

Therefore, by Corollary 4.14

$$\overline{\mathrm{H}}^{sep} = \mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e},$$

for some vector space decomposition $V = W_0 \oplus \cdots \oplus W_e$.

We need a technical lemma.

Lemma 5.1. *Let* H *be an unstable Noetherian integral domain over the Steenrod algebra. Then for all* $i \in \mathbb{N}_0$ *we have*

$$(\overline{\mathrm{H}}^{sep})_i = \overline{(\mathrm{H}_i)}^{sep}.$$

Proof. By induction on i, we need to prove the statement only for $i = 1$. By assumption we have the diagram

$$\begin{array}{ccccc} \mathrm{H} & \hookrightarrow & \mathrm{H}_1 & \hookrightarrow & \overline{(\mathbb{H}_1)}^{sep} \\ \downarrow & & & & \\ \overline{\mathbb{H}}^{sep} & \hookrightarrow & (\overline{\mathbb{H}}^{sep})_1. & & \end{array}$$

If $h \in \mathrm{H}_1$, then $h^p \in \mathrm{H} \subseteq \overline{\mathrm{H}}^{sep}$. Thus $h \in (\overline{\mathrm{H}}^{sep})_1$. Thus

$$\mathrm{H}_1 \hookrightarrow (\overline{\mathrm{H}}^{sep})_1.$$

We note that $\overline{\mathrm{H}}^{sep} = \mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e}$ by Corollary 4.14. Therefore

$$(\overline{\mathrm{H}}^{sep})_1 = \mathbb{F}[W_0 \oplus W_1] \otimes \mathbb{F}[W_2]^p \otimes \cdots \otimes \mathbb{F}[W_e]^{p^{e-1}}.$$

Hence $(\overline{\mathrm{H}}^{sep})_1$ is separably closed, and thus

$$\overline{(\mathrm{H}_1)}^{sep} \hookrightarrow (\overline{\mathrm{H}}^{sep})_1.$$

Moreover, this extension is by the universal property of the separable closure purely inseparable. Next we show that this extension has exponent at most one. To this end, take $h \in (\overline{\mathrm{H}}^{sep})_1$. Then $h^p \in \overline{\mathrm{H}}^{sep}$ is separable over H, hence over H_1. Therefore $h^p \in \overline{(\mathrm{H}_1)}^{sep}$.

Denote the inseparable closure of H_1 inside $(\overline{\mathrm{H}}^{sep})_1$ by K. Then $\mathrm{H}_1 \hookrightarrow \mathrm{K}$ has exponent at most one, and since $\overline{\mathrm{H}}^{sep} \hookrightarrow (\overline{\mathrm{H}}^{sep})_1$ has exponent one, the extension $\mathrm{H} \hookrightarrow \mathrm{K}$ also has exponent at most one. Since H_1 is the largest algebra such that $\mathrm{H} \hookrightarrow \mathrm{K}$ has exponent one, we have that $\mathrm{H}_1 = \mathrm{K}$ and $\mathrm{H}_1 \hookrightarrow (\overline{\mathrm{H}}^{sep})_1$ is separable.

Therefore, $\overline{(\mathrm{H}_1)}^{sep} \hookrightarrow (\overline{\mathrm{H}}^{sep})_1$ is also separable. Since we already saw that this extension is purely inseparable, this means that

$$(\overline{\mathrm{H}})^{sep})_1 = \overline{(\mathrm{H}_1)}^{sep}$$

as claimed. □

So, in what follows we can write $\overline{\mathrm{H}_1}^{sep}$ for $\overline{(\mathrm{H}_1)}^{sep} = (\overline{\mathrm{H}}^{sep})_1$ without ambiguity.

Theorem 5.2. *Let* H *be an integrally closed unstable Noetherian integral domain over the Steenrod algebra of Krull dimension* n. *Set* $\dim_{\mathbb{F}}(V) = n$. *Let* $V = W_0 \oplus \cdots \oplus W_e$*, and let*

$$\overline{\mathrm{H}}^{sep} = \mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e}.$$

Then there exists a group $G \leq \mathrm{GL}(V)$ *acting on the flags* $W_0 \oplus \cdots \oplus W_i$ *for* $i = 0, \ldots, e$ *such that* $\sqrt[\mathcal{P}^*]{\mathrm{H}} = \mathbb{F}[V]^G$ *and*

$$\mathrm{H} = (\mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e})^G.$$

Furthermore, $\dim_{\mathbb{F}}(W_0 \oplus \cdots \oplus W_i)$ *is the* Δ*-length of* H_i.

Proof. By assumption we have a diagram

$$\begin{array}{ccc} \mathrm{H} & \hookrightarrow & \sqrt[\mathcal{P}^*]{\mathrm{H}} = \mathbb{F}[V]^G \\ \big\downarrow & & \big\downarrow \\ \overline{\mathrm{H}}^{sep} & \hookrightarrow & \mathbb{F}[V] \end{array}$$

where the horizontal extensions are purely inseparable and the vertical are separable. Recall that

$$\overline{\mathrm{H}}^{sep} = \mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e}$$

where $\dim_{\mathbb{F}}(W_0 \oplus \cdots \oplus W_i)$ is the Δ-length of $\overline{\mathrm{H}_i}^{sep}$ by Corollary 4.14. Consider the corresponding diagram of the respective field of fractions

$$\begin{array}{ccc} \mathbb{H} & \hookrightarrow & \mathbb{F}(V)^G \\ \big\downarrow & & \big\downarrow \\ \overline{\mathbb{H}}^{sep} & \hookrightarrow & \mathbb{F}(V). \end{array}$$

Recall from the Imbedding Theorem (Theorem 8.1.5 in [3]) that H, and hence $\mathbb{H}$, contains a fractal of the Dickson algebra in dimension $n = \dim_{\mathbb{F}}(V)$. Thus

$$\mathcal{D}(n)^{q^s} \hookrightarrow \mathrm{H} \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{F}(V)$$

for some $s \in \mathrm{N}_0$. Therefore, the polynomial

$$\Delta(X) = \prod_{l \in V^*} (X - l)^{q^s} = \mathbf{d}_{n,0}^{q^s} X^{q^s} - \mathbf{d}_{n,1}^{q^s} X^{q^{s+1}} + \cdots + (-1)^n \mathbf{d}_{n,n}^{q^s} X^{q^{n+s}}$$

has coefficients in $\mathbb{H}$ (cf. Section 5.1 in [3]). Its roots are by construction the linear forms in $\mathbb{F}[V]$. Thus $\mathbb{F}(V)$ is the splitting field of $\Delta(X)$. Hence, the field extension $\mathbb{H} \hookrightarrow \mathbb{F}(V)$ is normal.[7] Since $\overline{\mathbb{H}}^{sep} \hookrightarrow \mathbb{F}(V)$ is purely inseparable, it follows from the structure theorem for finite dimensional normal field extensions that the extension

$$\mathbb{H} \hookrightarrow \overline{\mathbb{H}}^{sep}$$

is Galois with some Galois group G'. We have

$$|G'| = |\overline{\mathbb{H}}^{sep} \colon \mathbb{H}| = |\mathbb{F}(V) \colon \mathbb{F}(V)^G| = |G|.$$

Since

$$\mathbb{H} = (\overline{\mathbb{H}}^{sep})^{G'} = \overline{\mathbb{H}}^{sep} \cap \mathbb{F}[V]^G$$

[7]Note that this means that $\mathbb{F}(V)$ is algebraically closed in the category of fields over the Steenrod algebra; cf. Section 3.2 in [3].

it follows that $G' \geq G$. Thus $G' = G$. Finally we show that the Δ-length of an unstable algebra H coincides with the Δ-length of its separable closure ($\overline{\mathrm{H}}^{sep}$). Together with Lemma 5.1 this gives the result.

To this end, let $l_i = \dim_{\mathbb{F}}(W_i)$. Since G acts on $\overline{\mathrm{H}}^{sep}$, the group G consists of matrices of the form

$$(+) \qquad \begin{bmatrix} A_0 & 0 & & \dots & 0 \\ * & A_1 & 0 & \dots & 0 \\ & * & \ddots & & \vdots \\ \dots & & \ddots & & 0 \\ * & & \dots & * & A_e \end{bmatrix}$$

where A_i is an $n_i \times n_i$-matrix with $n_i = \dim(W_i)$. Denote by $\widehat{G}$ the subgroup of $\mathrm{GL}(n, \mathbb{F})$ consisting of *all* matrices of the form (+). Denote by $x_1, \dots, x_n$ a basis for $W_0 \oplus \cdots \oplus W_e$. Then

$$\begin{aligned} &(\mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e})^{\widehat{G}} \\ &\quad = \mathcal{D}(n_0) \otimes \mathbb{F}[c_{\mathrm{top}}(x^p_{n_0+1}), \dots, c_{\mathrm{top}}(x^p_{n_1}), c_{\mathrm{top}}(x^{p^2}_{n_1+1}), \dots, c_{\mathrm{top}}(x^{p^e}_n), \end{aligned}$$

where $c_{\mathrm{top}}(-)$ denotes the top orbit Chern class of the element $-$ (cf. Section 4.1 in [6]). By construction, the top orbit Chern classes of pth powers are pth powers. Thus the Δ-length of the ring of invariants $\widehat{G}$ is equal to n_0. Therefore we have

$$\begin{aligned} (\mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e})^{\widehat{G}} &\hookrightarrow \mathrm{H} = (\mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e})^{G} \\ &\hookrightarrow \overline{\mathrm{H}}^{sep} = \mathbb{F}[W_0] \otimes \cdots \mathbb{F}[W_e]^{p^e}. \end{aligned}$$

The smallest algebra, as well as the largest algebra in this chain, has Δ-length n_0. Thus by Lemma 3.7 we are done. □

Remark. Since G acts on $\overline{\mathrm{H}}^{sep}$, the group G consists of matrices of the form given in (+). So, if there exists no basis such that G consists of flag matrices like above, then the only unstable algebras $\mathrm{H} \hookrightarrow \sqrt[\mathcal{P}^*]{\mathrm{H}} = \mathbb{F}[V]^G$ are the p^sth powers

$$\mathrm{H} = (\mathbb{F}[V]^G)^{p^s},$$

i.e., we have the trivial vector space decomposition $V = W_e$.

Remark. Note carefully that the proof shows that the Δ-length of H and the Δ-length of any separable extension $\mathrm{H} \hookrightarrow \mathrm{K}$ coincide.

Remark. In Theorem 7.2.2 in [3] as well as in Theorem II in [8] it has been proven that

$$\mathrm{H} = (\mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e})^G.$$

However, the precise statement on the dimensions of $W_0 \oplus \cdots \oplus W_i$ is missing. Also the connection between the two Galois groups of $\mathrm{H} \hookrightarrow \mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e}$, resp. $\sqrt[\mathcal{P}^*]{\mathrm{H}} \hookrightarrow \mathbb{F}[V]$, is not made.

We conclude this section with an example.

Example 5.3. Consider the regular representation of the cyclic group of order 2, $\mathbb{Z}/2$, over a field of characteristic 2 afforded by the matrix

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Set $\mathbb{F}[V] = \mathbb{F}[x, y]$. Its ring of invariants is

$$\mathbb{F}[x, y]^{\mathbb{Z}/2} = \mathbb{F}[x, y^2 - yx].$$

Then $\mathbb{Z}/2$ acts on $\mathbb{F}[x, y^2]$ with invariant ring

$$\mathbb{F}[x, y^2]^{\mathbb{Z}/2} = \mathbb{F}[x, (y^2 - yx)^2].$$

On the other hand, $\mathbb{Z}/2$ does not act on $\mathbb{F}[x^2, y]$. Indeed

$$\mathbb{F}[x, y]^{\mathbb{Z}/2} \cap \mathbb{F}[x^2, y] = \mathbb{F}[x^2, (y^2 - yx)^2] = \mathbb{F}[x^2, y^2]^{\mathbb{Z}/2}.$$

Furthermore we could consider the purely inseparable field extension

$$\mathbb{F}(x^2, y^2 - yx) \hookrightarrow \mathbb{F}(x, y^2 + xy)$$

of degree 2. Note that

$$(*) \qquad \mathscr{P}^1(y^2 + xy) = x^2y + xy^2 \notin \mathbb{F}(x^2, y^2 - yx)$$

since our field contains only elements of even degree. Thus $\mathbb{F}(x^2, y^2 - yx)$ is not a field over the Steenrod algebra. Its separable closure

$$\mathbb{F}(x^2, y^2, yx) = \mathbb{F}\left(\frac{x}{y}, y^2\right)$$

is a Galois extension with the same Galois group $\mathbb{Z}/2$. However, the same calculation as above shows that it is also not closed under the action of the Steenrod algebra, as predicted in the previous result. Indeed, $\mathbb{F}(x, y)$ is the smallest overfield of $\mathbb{F}(x^2, y^2, yx)$, say $\mathbb{K}$, closed under the action of the Steenrod algebra as we see next:

$$\mathscr{P}^1(xy) = x^2y + xy^2 = xy(x + y) \in \mathbb{K} \Rightarrow x + y \in \mathbb{K}.$$

Since $\mathbb{K}$ must have the form $\mathbb{F}(W) \otimes \mathbb{F}(V/W)^2$ for some $W \leq V$ we find that $\operatorname{span}_{\mathbb{F}}\{x + y\} \subseteq W$. The minimal polynomial of $x + y \in \mathbb{K}$ over $\mathbb{F}(x^2, y^2, xy)$,

$$p(X) = X^2 + (x^2 + y^2),$$

is inseparable of degree 2. Therefore

$$2 = |\mathbb{F}(x, y) \colon \mathbb{F}(x^2, y^2, xy| = |\mathbb{F}(x, y) \colon \mathbb{K}|\,|\mathbb{K} \colon \mathbb{F}(x^2, y^2, xy)| = 2|\mathbb{F}(x, y) \colon \mathbb{K}|,$$

and hence $\mathbb{F}(x, y) = \mathbb{K}$ as claimed.

On the other hand, the largest subfield, call in $\mathbb{L}$, of $\mathbb{F}(x^2, y^2, xy)$ that is closed under the Steenrod algebra is $\mathbb{F}(x^2, y^2)$: by Equation $(*)$ the field $\mathbb{L}$ does not contain xy. Since xy is the root of

$$p(X) = X^2 + (xy)^2 \in \mathbb{F}(x^2, y^2)[X]$$

we find that

$$2 = |\mathbb{F}(x^2, y^2, xy) \colon \mathbb{F}(x^2, y^2)| = |\mathbb{F}(x^2, y^2, xy) \colon \mathbb{L}|\,|\mathbb{L} \colon \mathbb{F}(x^2, y^2)| = 2|\mathbb{L} \colon \mathbb{F}(x^2, y^2)|$$

and hence $\mathbb{L} = \mathbb{F}(x^2, y^2)$.

6. Projective dimension

The goal of this section is to prove that a Noetherian reduced unstable algebra H is Cohen-Macaulay if and only if its inseparable closure $\sqrt[\mathcal{P}^*]{\mathrm{H}}$ is Cohen-Macaulay.

Let H be an unstable algebra over the Steenrod algebra $\mathcal{P}^*$. An ideal $I \subseteq \mathrm{H}$ is called $\mathcal{P}^*$-invariant if it is closed under the action of the Steenrod algebra.

Lemma 6.1. *Let* H *be an unstable algebra over the Steenrod algebra. For any* $s \in \mathbb{N}_0$*, the canonical inclusion*

$$\psi\colon \mathrm{H}^{p^s} \hookrightarrow \mathrm{H}$$

induces a bijection

$$\psi^*\colon \mathcal{P}roj_{\mathcal{P}^*}(\mathrm{H}) \to \mathcal{P}roj_{\mathcal{P}^*}(\mathrm{H}^{p^s})$$

between the spaces of homogeneous $\mathcal{P}^*$*-invariant prime ideals.*

Proof. Since ψ is an integral extension, the Lying-Over Theorem holds. Thus ψ^* is surjective.

To prove injectivity take two homogeneous $\mathcal{P}^*$-invariant prime ideals $\mathfrak{p}_1, \mathfrak{p}_2 \subseteq \mathrm{H}$, such that

$$\mathfrak{p}_1 \cap \mathrm{H}^{p^s} = \mathfrak{p}_2 \cap \mathrm{H}^{p^s}.$$

Thus for any $h \in \mathfrak{p}_1$ it follows that

$$h^{p^s} \in \mathfrak{p}_1 \cap \mathrm{H}^{p^s} = \mathfrak{p}_2 \cap \mathrm{H}^{p^s}.$$

Therefore

$$h^{p^s} \in (\psi(\mathfrak{p}_2 \cap \mathrm{H}^{p^s})) \subseteq \mathfrak{p}_2.$$

Since $\mathfrak{p}_2$ is prime, we find that $h \in \mathfrak{p}_2$. Interchanging the roles of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ gives the result. $\square$

This result could have been proven also by observing that the sth iteration of the Frobenius map

$$F^s\colon \mathrm{H} \to \mathrm{H}^{p^s}$$

hands us an isomorphism of unstable algebras of degree p^s if H is reduced. This in turn also implies the following result.

Lemma 6.2. *Let* H *be an unstable reduced algebra over the Steenrod algebra. For any* $s \in \mathrm{N}_0$ *we find that*

$$\mathrm{depth}(\mathrm{H}) = \mathrm{depth}(\mathrm{H}^{p^s}).$$

$\square$

We observe that H is Noetherian of Krull dimension n if and only if H^{p^s} is. We find the following lemma.

Lemma 6.3. *Let* H *be Noetherian and reduced of Krull dimension* n*. Let* $S = \mathbb{F}[h_1^{p^s}, \dots, h_n^{p^s}]$ *be a system of parameters in* H^{p^s}*. Then*

$$\mathrm{proj}-\dim_S(\mathrm{H}) = \mathrm{proj}-\dim_S(\mathrm{H}^{p^s}) < \infty.$$

Proof. Since $\mathrm{H}^{p^s} \subseteq \mathrm{H}$ is a finite integral extension, S is also a system of parameters for H. Thus both projective dimensions are finite. Moreover, by the Auslander-Buchsbaum formula we have

$$\mathrm{proj}-\dim_S(\mathrm{H}^{p^s}) = \dim(\mathrm{H}^{p^s}) - \mathrm{depth}(\mathrm{H}^{p^s}) = \dim(\mathrm{H}) - \mathrm{depth}(\mathrm{H}) = \mathrm{proj}-\dim_S(\mathrm{H})$$

by Lemma 6.2. $\square$

We come to the desired result about a Noetherian unstable algebra H and its $\mathcal{P}^*$-inseparable closure $\sqrt[\mathcal{P}^*]{\mathrm{H}}$.

Proposition 6.4. *Let* H *be Noetherian and reduced of Krull dimension n. Then* H *is Cohen-Macaulay if and only if* $\sqrt[\mathcal{P}^*]{\mathrm{H}}$ *is Cohen-Macaulay.*

Proof. Since H is Noetherian, its $\mathcal{P}^*$-inseparable closure is also Noetherian by Theorem 6.1.3 in [3]. Therefore $\sqrt[\mathcal{P}^*]{\mathrm{H}} = \mathrm{H}_s$ for some $s \in \mathbb{N}_0$. Thus

$$(\sqrt[\mathcal{P}^*]{\mathrm{H}})^{p^s} \hookrightarrow \mathrm{H} \hookrightarrow \sqrt[\mathcal{P}^*]{\mathrm{H}} = \mathrm{H}_s$$

is a finite integral extension. By Lemma 6.1 we have a bijection

$$\mathcal{P}roj_{\mathcal{P}^*}(\sqrt[\mathcal{P}^*]{\mathrm{H}})^{p^s} \to \mathcal{P}roj_{\mathcal{P}^*}(\sqrt[\mathcal{P}^*]{\mathrm{H}}).$$

By Theorem 4.3.1 in [3] and Lemma 6.1

$$\mathcal{P}roj_{\mathcal{P}^*}(\sqrt[\mathcal{P}^*]{\mathrm{H}})^{p^s} \to \mathcal{P}roj_{\mathcal{P}^*}(\mathrm{H}) \to \mathcal{P}roj_{\mathcal{P}^*}(\sqrt[\mathcal{P}^*]{\mathrm{H}})$$

is also bijective. Moreover, by Lemma 6.2, the left and the right algebra have the same depth. Thus by Theorem 2.1 in [5] the results follows (cf. Corollary 2.2 loc.cit.). □

7. Polynomial rings

Let H be an integrally closed unstable Noetherian integral domain over the Steenrod algebra. By Theorem 5.2 we have

$$\mathrm{H} = (\overline{\mathrm{H}}^{sep})^G \hookrightarrow \sqrt[\mathcal{P}^*]{\mathrm{H}} = \mathbb{F}[V]^G,$$

where

$$\overline{\mathrm{H}}^{sep} = \mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e}$$

for some vector space decomposition $V = W_0 \oplus \cdots \oplus W_e$. By Proposition 6.4 we know that H is Cohen-Macaulay if and only if $\sqrt[\mathcal{P}^*]{\mathrm{H}}$ is polynomial. Moreover, the algebra generators of H are just suitable p^sth powers of the algebra generators of $\sqrt[\mathcal{P}^*]{\mathrm{H}}$ (for a minimal generating set).

Let G act on $V = W_0 \oplus \cdots \oplus W_e$ such that

$$gw_i \in W_0 \oplus \cdots \oplus W_i$$

for all $w_i \in W_i$, i.e., G consists of flag matrices of the form

$$\begin{bmatrix} A_0 & 0 & & \dots & 0 \\ * & A_1 & 0 & \dots & 0 \\ & * & \ddots & & \vdots \\ \dots & & \ddots & & 0 \\ * & & \dots & * & A_e \end{bmatrix}$$

where A_i is an $m_i \times m_i$-matrix with $m_i = \dim(W_i)$. For every $i = 0, \dots, e$ we have a group epimorphism

$$\mathrm{pr}_i \colon G \to G_i, \begin{bmatrix} A_0 & 0 & & \dots & 0 \\ * & A_1 & 0 & \dots & 0 \\ & * & \ddots & & \vdots \\ \dots & & \ddots & & 0 \\ * & & \dots & * & A_e \end{bmatrix} \mapsto \begin{bmatrix} A_0 & 0 & & \dots & 0 \\ * & A_1 & 0 & \dots & 0 \\ & * & \ddots & & \vdots \\ \dots & & \ddots & & 0 \\ * & & \dots & * & A_i \end{bmatrix}.$$

Lemma 7.1. *With the preceding notation we have*

$$\mathbb{F}[W_0 \oplus \cdots \oplus W_i]^{G_i} = \mathbb{F}[V]^G \cap \mathbb{F}[W_0 \oplus \cdots \oplus W_i] \subseteq \mathbb{F}[V]^G.$$

Proof. The kernel of the projection pr_i, $\ker(\mathrm{pr}_i)$, consists of matrices of the form

$$\begin{bmatrix} I_0 & 0 & & & \dots & 0 \\ 0 & \ddots & & & & 0 \\ \vdots & & I_i & 0 & \dots & 0 \\ * & * & * & A_{i+1} & & \vdots \\ & & & * & \ddots & \\ * & & \dots & * & & A_e \end{bmatrix},$$

where the I_j's are identity matrices. Thus $\mathbb{F}[V]^{\ker(\mathrm{pr}_i)} \supseteq \mathbb{F}[W_0 \oplus \cdots \oplus W_i]$, and hence

$$\mathbb{F}[V]^G = (\mathbb{F}[V]^{\ker(\mathrm{pr}_i)})^{G_i} \supseteq \mathbb{F}[W_0 \oplus \cdots \oplus W_i]^{G_i}.$$

Since $\mathbb{F}[W_0 \oplus \cdots \oplus W_i]^{G_i} \subseteq \mathbb{F}[W_0 \oplus \cdots \oplus W_i]$ we find

$$\mathbb{F}[W_0 \oplus \cdots \oplus W_i]^{G_i} \subseteq \mathbb{F}[V]^G \cap \mathbb{F}[W_0 \oplus \cdots \oplus W_i] \subseteq \mathbb{F}[V]^G.$$

Conversely, since $\mathbb{F}[V]^{\ker(\mathrm{pr}_i)} \supseteq \mathbb{F}[W_0 \oplus \cdots \oplus W_i]$, we have

$$\mathbb{F}[V]^{\ker(\mathrm{pr}_i)} \cap \mathbb{F}[W_0 \oplus \cdots \oplus W_i] = \mathbb{F}[W_0 \oplus \cdots \oplus W_i].$$

Thus

$$\mathbb{F}[W_0 \oplus \cdots \oplus W_i]^{G_i} = (\mathbb{F}[V]^{\ker(\mathrm{pr}_i)} \cap \mathbb{F}[W_0 \oplus \cdots \oplus W_i])^{G_i}.$$

Finally, note that

$$\mathbb{F}[V]^G \cap \mathbb{F}[W_0 \oplus \cdots \oplus W_i] \subseteq (\mathbb{F}[V]^{\ker(\mathrm{pr}_i)} \cap \mathbb{F}[W_0 \oplus \cdots \oplus W_i])^{G_i}.$$

To see this, take an element $f \in \mathbb{F}[V]^G \cap \mathbb{F}[W_0 \oplus \cdots \oplus W_i]$. Then $f \in \mathbb{F}[W_0 \oplus \cdots \oplus W_i]$ is invariant under the group G. Thus f is also invariant under $\ker(\mathrm{pr}_i)$. Therefore,

$$f \in \mathbb{F}[V]^{\ker(\mathrm{pr}_i)} \cap \mathbb{F}[W_0 \oplus \cdots \oplus W_i].$$

But f is also G-invariant, i.e.,

$$f \in (\mathbb{F}[V]^{\ker(\mathrm{pr}_i)} \cap \mathbb{F}[W_0 \oplus \cdots \oplus W_i])^G \subseteq (\mathbb{F}[V]^{\ker(\mathrm{pr}_i)} \cap \mathbb{F}[W_0 \oplus \cdots \oplus w_i])^{G_i}$$

as desired. □

Let $h_1, \ldots, h_m \in \mathbb{F}[V]^G$ be a minimal generating set. Without loss of generality we assume that they are sorted such that

$$\begin{aligned} &h_1, \ldots, h_{n_0} \in \mathbb{F}[W_0], \\ &h_{n_0+1}, \ldots, h_{n_1} \in \mathbb{F}[W_0 \oplus W_1], \\ &\dots \\ &h_{n_{e-1}+1}, \ldots, h_{n_e} = h_m \in \mathbb{F}[W_0 \oplus \cdots \oplus W_e]. \end{aligned}$$

We assume that $n_0, \ldots, n_e$ are maximal with this property. Thus by construction

$$\mathbb{F}[h_0, \ldots, h_{n_i}] \subseteq \mathbb{F}[W_0 \oplus \cdots \oplus W_i]^{G_i} \subseteq \mathbb{F}[V]^G$$

for all $i = 0, \ldots, e$.

Proposition 7.2. *If $n_e = \dim_{\mathbb{F}}(V)$, i.e., if the ring of invariants*

$$\mathbb{F}[W_0 \oplus \cdots \oplus W_e]^G$$

is polynomial, then $n_i = \dim_{\mathbb{F}}(W_0 \oplus \cdots \oplus W_i)$.

Proof. Consider the integral extension

$$\mathbb{F}[W_0 \oplus \cdots \oplus W_i]^{G_i} \hookrightarrow \mathbb{F}[W_0 \oplus \cdots \oplus W_i].$$

The maximal ideal $\mathfrak{m}_i$ of $\mathbb{F}[W_0 \oplus \cdots \oplus W_i]$ lies over the maximal ideal in $\mathbb{F}[W_0 \oplus \cdots \oplus W_i]^{G_i}$. Furthermore, $\mathfrak{m}_i$ extends to a prime ideal $\mathfrak{p}_i \subseteq \mathbb{F}[W_0 \oplus \cdots \oplus W_e]$. By construction $\mathfrak{p}_i$ is generated by all linear forms in $\mathbb{F}[W_0 \oplus \cdots \oplus W_i]$. Thus $\mathfrak{p}_i$ is regular and prime of height equal to $\dim_{\mathbb{F}}(W_0 \oplus \cdots \oplus W_i)$. Hence, its contraction to the ring of invariants

$$\mathfrak{p}_i^c = \mathfrak{p}_i \cap \mathbb{F}[W_0 \oplus \cdots \oplus W_e]^G$$

is also prime of height equal to $\dim_{\mathbb{F}}(W_0 \oplus \cdots \oplus W_i)$. Furthermore, $\mathfrak{p}_i^c$ contains by construction

$$(h_1, \ldots, h_{n_i}) \subseteq \mathfrak{p}_i^c.$$

Thus the quotient

$$\mathbb{F}[W_0 \oplus \cdots \oplus W_e]^G/\mathfrak{p}^c = \mathbb{F}[\overline{h}_{n_i+1}, \ldots, \overline{h}_n] \hookrightarrow \mathbb{F}[W_{i+1} \oplus \cdots \oplus W_e]$$

is integral, and

$$n - n_i = \dim_{\mathbb{F}}(W_{i+1} \oplus \cdots \oplus W_e)$$

for all $i = 0, \ldots, e-1$. □

Theorem 7.3. *With the above notation, if*

$$(\mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e})^G = \mathbb{F}[h_1, \ldots, h_n]$$

is polynomial, then for suitable $s_1, \ldots, s_n \in \mathbb{N}_0$

$$(\mathbb{F}[U_0] \otimes \cdots \otimes \mathbb{F}[U_f]^{p^f})^G = \mathbb{F}[h_1^{p^{s_1}}, \ldots, h_n^{p^{s_n}}]$$

is polynomial for any subflag

$$U_0 \oplus \cdots \oplus U_j \leq W_0 \oplus \cdots \oplus W_i$$

that admits an action of G.

Proof. To simplify notation we assume that the extension

$$\mathbb{F}[U_0] \otimes \cdots \otimes \mathbb{F}[U_f]^{p^f} \hookrightarrow \mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e}$$

is purely inseparable of exponent one. The general case follows then inductively.

Since G acts on the flag $W_0 \oplus \cdots \oplus W_e$ the algebra generator for the ring of invariants can be sorted such that

$$h_1, \ldots, h_{n_i} \in \mathbb{F}[W_0 \oplus \cdots \oplus W_i]$$

with $n_i - n_{i-1} = \dim(W_i)$, $n_0 = \dim_{\mathbb{F}}(W_0)$, by Proposition 7.2.

Since G acts also on the subflag $U_0 \oplus \cdots \oplus U_f$ the algebra generator for the ring of invariants can be sorted such that

$$h_1, \ldots, h_{m_i} \in \mathbb{F}[U_0 \oplus \cdots \oplus U_i]$$

with $m_i - m_{i-1} = \dim(U_i)$, $m_0 = \dim_{\mathbb{F}}(U_0)$, and $m_f = n_e = n$. Thus $n_i \geq m_i$. Consider the algebra

$$\begin{aligned} \mathrm{A} = \mathbb{F}[h_1, \dots, h_{m_0}, h_{m_0+1}^p, \dots, h_{n_0}^p, h_{n_0+1}, \dots, \\ h_{m_1}, h_{m_1+1}^p, \dots, h_{n_1}^p, \dots, h_{m_e}, h_{m_e+1}^p, \dots, h_{m_f}^p] \\ \hookrightarrow \mathbb{F}[U_0] \otimes \cdots \otimes \mathbb{F}[U_f]^{p^f}. \end{aligned}$$

Since A consists of invariant polynomials it is contained in the ring of invariants

$$(\mathbb{F}[U_0] \otimes \cdots \otimes \mathbb{F}[U_f]^{p^f})^G.$$

The diagram

$$\begin{array}{ccccc} A & \hookrightarrow & (\mathbb{F}[U_0] \otimes \cdots \otimes \mathbb{F}[U_f]^{p^f})^G & \hookrightarrow & \mathbb{F}[U_0] \otimes \cdots \otimes \mathbb{F}[U_f]^{p^f} \\ \downarrow & & & & \downarrow \\ \mathbb{F}[h_1, \dots, h_n] & & \hookrightarrow & & \mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e} \end{array}$$

has by construction purely inseparable vertical extensions of degree $p^{\sum_i (n_i - m_i)}$. Since the degree of

$$\mathbb{F}[h_1, \dots, h_n] \hookrightarrow \mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e}$$

is the group order $|G|$, the degree of

$$A \hookrightarrow \mathbb{F}[U_0] \otimes \cdots \otimes \mathbb{F}[U_f]^{p^f}$$

is also the group order. Thus A is the desired ring of invariants as claimed. □

The following result settles a twenty-year-old conjecture due to Clarence W. Wilkerson (see Conjecture 5.1 in [8]).

Theorem 7.4. *Let* H *be an integrally closed Noetherian unstable integral domain over the Steenrod algebra. Then* H *is polynomial if and only if* ${}^{\mathcal{P}^*}\!\sqrt{\mathrm{H}}$ *is polynomial. Furthermore,*

$${}^{\mathcal{P}^*}\!\sqrt{\mathrm{H}} = \mathbb{F}[h_1, \dots, h_n]$$

if and only if there are $s_1, \dots, s_n \in \mathbb{N}_0$ *such that*

$$\mathrm{H} = \mathbb{F}[h_1^{p^{s_1}}, \dots, h_n^{p^n}].$$

Proof. By Theorem 5.2 there exist a group G and a flag $V = W_0 \oplus \cdots \oplus W_e$ such that

$$\mathrm{H} = (\mathbb{F}[W_0] \otimes \cdots \otimes \mathbb{F}[W_e]^{p^e})^G \hookrightarrow {}^{\mathcal{P}^*}\!\sqrt{\mathrm{H}} = \mathbb{F}[V]^G.$$

If ${}^{\mathcal{P}^*}\!\sqrt{\mathrm{H}}$ is polynomial, then so is H by Theorem 7.3. Note that the same result also gives the precise statement on the respective algebra generators.

On the other hand, $({}^{\mathcal{P}^*}\!\sqrt{\mathrm{H}})^{p^e} = (\mathbb{F}[V]^{p^e})^G \hookrightarrow \mathrm{H}$ is the ring of invariants on the subflag $V \leq W_0 \oplus \cdots \oplus W_e$ for some large enough e. Therefore if H is polynomial, then $({}^{\mathcal{P}^*}\!\sqrt{\mathrm{H}})^{p^e} \hookrightarrow \mathrm{H}$ is polynomial by the same Theorem 7.3. Thus ${}^{\mathcal{P}^*}\!\sqrt{\mathrm{H}}$ is polynomial since it is isomorphic as an algebra to $({}^{\mathcal{P}^*}\!\sqrt{\mathrm{H}})^{p^e}$. □

Thus we have the following corollary.

Corollary 7.5. *Let* H *be an unstable polynomial algebra over the Steenrod algebra. Set* $\mathrm{H} = \mathbb{F}[h_1, \dots, h_n]$. *Then* H *is* $\mathcal{P}^*$*-inseparably closed if and only if the polynomial generators* $h_1, \dots, h_n$ *are no pth powers.* □

The example given at the end of Section 5 illustrates these results. We want to close with an example that shows that a simple generalization of Theorem 7.4 to nonpolynomial invariants is not true.

Example 7.6. Let p be odd and let $\mathbb{F}$ be the prime field of characteristic p. Consider the four-dimensional modular representation $\mathbb{Z}/p \hookrightarrow \mathrm{GL}(4, \mathbb{F})$ afforded by the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Its ring of invariants turns out to be a hypersurface

$$\mathbb{F}[x_1, y_1, x_2, y_2]^{\mathbb{Z}/p} = \mathbb{F}[c_1, y_1, c_2, y_2, q]/(r),$$

where $c_i = x_i^p - x_i y_i^{p-1}$ are the top orbit Chern classes of x_i, $i = 1, 2$, and $q = x_1 y_2 - x_1 y_1$ is an invariant quadratic form. The relation is given by

$$r = q^p - c_1 y_2^p + c_2 y_1^p + q y_1^{p-1} y_2^{p-1}$$

(see Theorem 2.1 in [2]). Certainly, $\mathbb{Z}/p$ also acts on $\mathbb{F}[x_1, y_1] \otimes \mathbb{F}[x_2^p, y_2^p]$ and we find that

$$\mathrm{A} = \mathbb{F}[c_1, y_1, c_2^p, y_2^p, q^p] \hookrightarrow (\mathbb{F}[x_1, y_1] \otimes \mathbb{F}[x_2^p, y_2^p])^{\mathbb{Z}/p}.$$

However, the new ring of invariants contains an invariant that is not in the algebra A, namely

$$q' = x_1 y_2^p - x_2^p y_1.$$

Indeed, with the methods presented in Theorem 2.1 of [2] it is not hard to see that

$$(\mathbb{F}[x_1, y_1] \otimes \mathbb{F}[x_2^p, y2^p])^{\mathbb{Z}/p} = \mathbb{F}[c_1, y_1, c_2^p, y_2^p, q']/(r'),$$

where $r' = (q')^p - c_1 y_2^{p^2} + c_2^p y_1^p - q' y_1^{p-1} y_2^{p(p-1)}$. Interesting enough though, it transpires that this ring is again a hypersurface.

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